ACOUSTIC PRESSURE OSCILLATION IN A PIPE VIA THE FINITE ELEMENT METHOD Revision B

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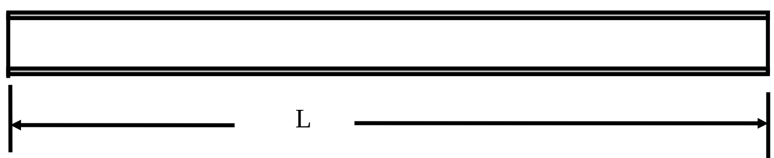
November 13, 2008

Introduction

The purpose of this tutorial is to derive for a method for analyzing the acoustic pressure oscillation in a pipe using the finite element method. The method is based on Reference 1.

Theory

Consider the pipe in Figure 1, where the length is much greater than the diameter. The cross-section may have an arbitrary shape. Assume that the pipe is filled with a gas.



L is the length.
c is the speed of sound in the enclosed gas

Figure 1.

The acoustic modes of the pipe can be found by classical methods.

The natural frequency equation of either an open-open or a closed-closed pipe is

$$f_i = \frac{c}{2L}, \quad i = 1, 2, 3, \ldots \tag{1}$$

The natural frequency equation of a closed-open pipe is

$$f_i = \frac{c}{4L}, \quad i = 1, 3, 5, \ldots \tag{2}$$

The natural frequencies are independent of the cross-section assuming that the length is much greater than the diameter.

Let p(x,t) represent the pressure of the pipe as a function of space and time.

The free, transverse vibration of the pipe is governed by the equation:

$$c^2 \frac{d^2}{\partial x^2} p(x,t) = \frac{\partial^2 p(x,t)}{\partial t^2} \tag{3}$$

Equation (3) is independent of the boundary conditions, which are applied as constraint equations.

Assume that the solution of equation (3) is separable in time and space.

$$p(x,t) = P(x) f(t) \tag{4}$$

$$c^2 \frac{\partial^2}{\partial x^2} P(x) f(t) = \frac{\partial^2 P(x) f(t)}{\partial t^2} \tag{5}$$

$$c^2 f(t) \frac{\partial^2}{\partial x^2} P(x) = P(x) \frac{\partial^2 f(t)}{\partial t^2} \tag{6}$$

The partial derivatives change to ordinary derivatives.

$$c^2 f(t) \frac{d^2}{d x^2} P(x) = P(x) \frac{d^2 f(t)}{d t^2} \tag{7}$$

$$c^2 \frac{1}{P(x)} \frac{d^2}{d x^2} P(x) = \frac{1}{f(t)} \frac{d^2 f(t)}{d t^2} \tag{8}$$

The left-hand side of equation (8) depends on x only. The right hand side depends on t only. Both x and t are independent variables. Thus equation (8) only has a solution if both sides are constant. Let $-\omega^2$ be the constant.

$$c^2 \frac{1}{P(x)} \frac{d^2}{d x^2} P(x) = \frac{1}{f(t)} \frac{d^2 f(t)}{d t^2} = -\omega^2 \tag{9}$$

Equation (9) yields two independent equations.

$$\frac{d^2}{dx^2} P(x) + \left(\frac{\omega}{c}\right)^2 P(x) = 0 \tag{10}$$

$$\frac{d^2}{dt^2} f(t) + \omega^2 f(t) = 0 \tag{11}$$

Equation (10) is a homogeneous, second order, ordinary differential equation.

The weighted residual method is applied to equation (10). This method is suitable for boundary value problems. An alternative method would be the energy method.

There are numerous techniques for applying the weighted residual method. Specifically, the Galerkin approach is used in this tutorial.

The differential equation (10) is multiplied by a test function $\phi(x)$. Note that the test function $\phi(x)$ must satisfy the homogeneous essential boundary conditions. The essential boundary conditions are the prescribed values of p and its first derivative.

The test function is not required to satisfy the differential equation, however.

The product of the test function and the differential equation is integrated over the domain. The integral is set equation to zero.

$$\int \phi(x) \left\{ \frac{d^2}{dx^2} P(x) + \left(\frac{\omega}{c}\right)^2 P(x) \right\} dx = 0 \tag{12}$$

The test function $\phi(x)$ can be regarded as a virtual pressure. The differential equation in the brackets represents an internal force. This term is also regarded as the residual. Thus, the integral represents virtual work, which should vanish at the equilibrium condition.

Define the domain over the limits from a to b. These limits represent the boundary points of the entire pipe.

$$\int_a^b \phi(x) \left\{ \frac{d^2}{dx^2} P(x) + \left(\frac{\omega}{c}\right)^2 P(x) \right\} dx = 0 \tag{13}$$

$$\int_a^b \phi(x) \left\{ \frac{d^2}{dx^2} P(x) \right\} dx + \int_a^b \phi(x) \left\{ \left(\frac{\omega}{c}\right)^2 P(x) \right\} dx = 0 \tag{14}$$

$$\int_a^b \phi(x)\left\{\frac{d^2}{dx^2}P(x)\right\}dx + \left(\frac{\omega}{c}\right)^2\int_a^b \phi(x)\{P(x)\}dx = 0 \tag{15}$$

Integrate the first integral by parts.

$$\int_a^b \frac{d}{dx}\left\{\phi(x)\frac{d}{dx}P(x)\right\}dx - \int_a^b\left\{\frac{d}{dx}\phi(x)\right\}\left\{\frac{d}{dx}P(x)\right\}dx$$

$$+\left(\frac{\omega}{c}\right)^2\int_a^b \phi(x)\{P(x)\}dx = 0 \tag{16}$$

$$\left\{\phi(x)\frac{d}{dx}P(x)\right\}\Bigg|_a^b - \int_a^b\left\{\frac{d}{dx}\phi(x)\right\}\left\{\frac{d}{dx}P(x)\right\}dx$$

$$+\left(\frac{\omega}{c}\right)^2\int_a^b \phi(x)\{P(x)\}dx = 0 \tag{17}$$

Consider a closed-open pipe. The boundary conditions are

$$\left.\frac{dP}{dx}\right|_{x=a} = 0 \tag{18}$$

$$P(b) = 0 \tag{19}$$

Thus, the test functions must satisfy

$$\left.\frac{d\phi}{dx}\right|_{x=a} = 0 \tag{20}$$

$$\phi(b) = 0 \tag{21}$$

Equations (20) and (21) require

$$\left\{ \phi(x)\,\frac{d}{dx}P(x) \right\}\Bigg|_a^b = 0 \tag{22}$$

Apply the boundary conditions to equation (17). The result is

$$-\int_a^b \left\{ \frac{d}{dx}\phi(x) \right\}\left\{ \frac{d}{dx}P(x) \right\} dx + \left(\frac{\omega}{c}\right)^2 \int_a^b \phi(x)\,\{P(x)\}\,dx = 0 \tag{23}$$

$$\int_a^b \left\{ \frac{d}{dx}\phi(x) \right\}\left\{ \frac{d}{dx}P(x) \right\} dx - \left(\frac{\omega}{c}\right)^2 \int_a^b \phi(x)\,\{P(x)\}\,dx = 0 \tag{24}$$

Note that equation (24) would also be obtained for other simple boundary condition cases.

Now consider that the pipe consists of number of segments, or elements. The elements are arranged geometrically in series form. Furthermore, the endpoints of each element are called nodes.

The following equation must be satisfied for each element.

$$\int \left\{ \frac{d}{dx}\phi(x) \right\}\left\{ \frac{d}{dx}P(x) \right\} dx - \left(\frac{\omega}{c}\right)^2 \int \phi(x)\,\{P(x)\}\,dx = 0 \tag{25}$$

The essence of the Galerkin method is that the test function is chosen as

$$\phi(x) = P(x) \tag{26}$$

Thus

$$\int \left\{ \frac{d}{dx}P(x) \right\}\left\{ \frac{d}{dx}P(x) \right\} dx - \left(\frac{\omega}{c}\right)^2 \int \{P(x)\}^2\,dx = 0 \tag{27}$$

Express the pressure function P(x) in terms of nodal pressures p_{j-1} and p_j .

$$P(x) = L_1\, p_{j-1} + L_2\, p_j \;, \qquad (j-1)h \le x \le jh \tag{28}$$

Note that h is the element length. In addition, each L coefficients is a function of x.

Now introduce a nondimensional natural coordinate ξ .

$$\xi = j - x/h \tag{29}$$

$$h\,\xi = h\,j - x \tag{30}$$

$$x = h\,j - h\,\xi \tag{31}$$

$$\left(\frac{x}{h}\right) = j - \xi \tag{32}$$

The derivative is

$$dx = -\,h\,d\xi \tag{33}$$

$$d\xi = -\frac{1}{h}dx \tag{34}$$

Note that h is the segment length.

Change the integration variable in equation (27) using equation (33). Also, apply the integration limits.

$$-h\int_0^1 \left\{\frac{d}{dx}P(x)\right\}\left\{\frac{d}{dx}P(x)\right\} d\xi + h\left(\frac{\omega}{c}\right)^2 \int_0^1 \{P(x)\}^2\, d\xi = 0 \tag{35}$$

$$h\int_0^1 \left\{\frac{d}{dx}P(\xi)\right\}\left\{\frac{d}{dx}P(\backslash\xi)\right\} d\xi - h\left(\frac{\omega}{c}\right)^2 \int_0^1 \{P(\xi)\}^2\, d\xi = 0 \tag{36}$$

The pressure function becomes.

$$P(\xi) = L_1\, p_{j-1} + L_2\, p_j \;, \qquad 0 \le \xi \le 1 \tag{37}$$

The slope equation is

$$P\,'(\xi) = L_1'\, y_{j-1} + L_2'\, h\,\theta_{j-1} \;, \qquad 0 \le \xi \le 1 \tag{38}$$

$$L_1 = 1 - \xi \tag{39}$$

$$L_1' = -1 \tag{40}$$

$$L_2 = \xi \tag{41}$$

$$L_2' = 1 \tag{42}$$

Now Let

$$P(x) = \underline{L}^{T}\;\bar{a}\,, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{43}$$

where

$$\underline{L} = [1-\xi \qquad \xi\;]^{T} \tag{44}$$

$$\bar{a} = [\;\; p_{j-1} \qquad p_j \;\;]^{T} \tag{45}$$

The derivative terms are

$$\frac{d}{dx}P(x) = \frac{d}{dx}\underline{L}^{T}\;\bar{a}\,, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{46}$$

$$\frac{d}{dx}P(x) = \frac{d}{d\xi}\frac{d\xi}{dx}\underline{L}^{T}\;\bar{a}\,, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{47}$$

$$\frac{d}{dx}P(x) = \left(-\frac{1}{h}\right)\underline{\underline{L}'^{\,T}}\ \bar{a}\,, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{48}$$

where

$$\underline{L}' = \begin{bmatrix} -1 & 1 \end{bmatrix}^T \tag{49}$$

Note that primes indicate derivatives with respect to ξ.

(50)

Equation (36) becomes

$$h\int_0^1 \left\{ \left[\left(\frac{1}{-h}\right) \underline{L}'^{\,T}\ \bar{a}\right]\left[\left(\frac{1}{-h}\right) \underline{L}'^{\,T}\ \bar{a}\right] \right\} d\xi$$

$$+h\left(\frac{\omega}{c}\right)^2 \int_0^1 \left[\ \underline{L}^T\ \bar{a}\right]\left[\underline{L}^T\ \bar{a}\right]\ d\xi = 0 \tag{51}$$

$$\int_0^1 \left\{ \left[\ \underline{L}'^{\,T}\ \bar{a}\right]\left[\ \underline{L}'^{\,T}\ \bar{a}\right] \right\} d\xi\ - h^2\left(\frac{\omega}{c}\right)^2 \int_0^1 \left[\ \underline{L}^T\ \bar{a}\right]\left[\underline{L}^T\ \bar{a}\right]\ d\xi = 0 \tag{52}$$

$$\int_0^1 \left\{ \bar{a}^T\, \underline{L}'\ \underline{L}'^{\,T}\ \bar{a} \right\} d\xi\ - \left(\frac{h\,\omega}{c}\right)^2 \int_0^1 \left[\bar{a}^T\, \underline{L}\, \underline{L}^T\ \bar{a}\right]\ d\xi = 0 \tag{53}$$

$$\bar{a}^T \left\{ \int_0^1 \left\{ \underline{L}'\ \underline{L}'^{\,T} \right\} d\xi\ - \left(\frac{h\,\omega}{c}\right)^2 \int_0^1 \left[\ \underline{L}\, \underline{L}^T\right]\ d\xi \right\} \bar{a}\ = 0 \tag{54}$$

$$\int_0^1 \left\{ \underline{L}'\ \underline{L}'^{\,T} \right\} d\xi\ - \left(\frac{h\,\omega}{c}\right)^2 \int_0^1 \left[\ \underline{L}\, \underline{L}^T\right]\ d\xi = 0 \tag{55}$$

For a system of n elements,

$$K_j - \lambda\, M_j = 0\,, \qquad j = 1, 2, \ldots, n \tag{56}$$

where

$$K_j \;=\; \int_0^1 \left\{ \underline{L}' \;\underline{L}'^{\,T} \right\} d\xi \tag{57}$$

$$M_j = \int_0^1 \left\{ \underline{L}\,\underline{L}^T \right\} d\xi \tag{58}$$

$$\lambda = \left(\frac{h\ \omega}{c} \right)^2 \tag{59}$$

$$\underline{L}'\,\underline{L}'^{T} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} \tag{60}$$

$$\underline{L}'\,\underline{L}'^{T} = \begin{bmatrix} 1 & -1 \\ & 1 \end{bmatrix} \tag{61}$$

Note that only the upper triangular components are shown due to symmetry.

$$K_j \;=\; \int_0^1 \begin{bmatrix} 1 & -1 \\ & 1 \end{bmatrix} d\xi \tag{62}$$

$$K_j \;=\; \left. \begin{bmatrix} \xi & -\xi \\ & \xi \end{bmatrix} \right|_0^1 \tag{63}$$

$$K_j \;=\; \begin{bmatrix} 1 & -1 \\ & 1 \end{bmatrix} \tag{64}$$

$$\underline{L}\ \underline{L}^T = \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \begin{bmatrix} 1-\xi & \xi \end{bmatrix} \tag{65}$$

$$\underline{L}\ \underline{L}^T = \begin{bmatrix} 1-2\xi+\xi^2 & \xi-\xi^2 \\ & \xi^2 \end{bmatrix} \tag{66}$$

Recall

$$M_j = \int_0^1 \left\{ \underline{L}\ \underline{L}^T \right\} d\xi \tag{67}$$

$$M_j = \int_0^1 \begin{bmatrix} 1-2\xi+\xi^2 & \xi-\xi^2 \\ & \xi^2 \end{bmatrix} d\xi \tag{68}$$

$$M_j = \left. \begin{bmatrix} \xi-\xi^2+\frac{1}{3}\xi^3 & \frac{1}{2}\xi^2-\frac{1}{3}\xi^3 \\ & \frac{1}{3}\xi^3 \end{bmatrix} \right|_0^1 \tag{69}$$

$$M_j = \begin{bmatrix} 1-1+\frac{1}{3} & \frac{1}{2}-\frac{1}{3} \\ & \frac{1}{3} \end{bmatrix} \tag{70}$$

$$M_j = \begin{bmatrix} 1/3 & 1/6 \\ & \\ & 1/3 \end{bmatrix} \tag{71}$$

$$M_j = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} \tag{72}$$

Examples are given in Appendices A and B.

An alternate derivation based on virtual work is given in Appendix C.

<u>References</u>

1. L. Segerlind, Applied Finite Element Analysis, Wiley, New York, 1984.
2. T. Irvine, The Generalized Eigenvalue Problem, 1999.
3. K. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, New Jersey, 1982.
4. W. Weaver and P. Johnson, Structural Dynamics by Finite Elements, Englewood Cliffs, New Jersey, 1987.

APPENDIX A

Example 1: Closed-Closed Pipe, FE Model, Two Elements

The finite element model of the pipe is shown in Figure A-1. It consists of two elements and three nodes. The pipe has length L. Each element has an equal length.

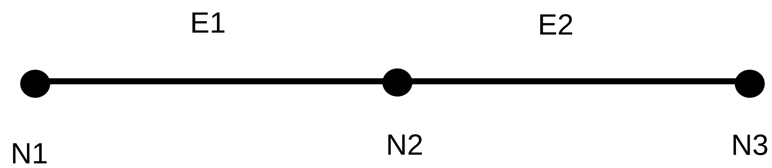


Figure A-1.

The boundary conditions are

$$\left.\frac{dP}{dx}\right|_{x=0} = 0 \qquad \text{(A-1)}$$

$$\left.\frac{dP}{dx}\right|_{x=L} = 0 \qquad \text{(A-2)}$$

The generalized eigenvalue problem is

$$K_j - \lambda\, M_j = 0\,, \qquad j = 1, 2, \ldots, n \qquad \text{(A-3)}$$

The elemental mass matrix is

$$M_j = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} \qquad \text{(A-4)}$$

The elemental stiffness matrix is

$$K_j = \begin{bmatrix} 1 & -1 \\ & 1 \end{bmatrix} \qquad \text{(A-5)}$$

The generalized eigenvalue problem with global mass and stiffness matrices is

$$\det\left\{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \left(\frac{\lambda}{6}\right)\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right\} = 0 \qquad \text{(A-7)}$$

Note that

$$\lambda = \left(\frac{h\ \omega}{c}\right)^2 \qquad \text{(A-6)}$$

$$h = L / 2 \qquad \text{(A-8)}$$

The eigenvalue problem is solved using the methods in References 2 and 3. The results can also be obtained via Matlab as follows.

```
k =

     1    -1     0
    -1     2    -1
     0    -1     1

m =

    0.3333    0.1667         0
    0.1667    0.6667    0.1667
         0    0.1667    0.3333

>> [ModeShapes,Eigenvalues]=eig(k,m);
>> Eigenvalues

Eigenvalues =

   -0.0000         0         0
         0    3.0000         0
         0         0   12.0000
```

The resulting natural frequencies and mode shapes are shown in Table A-1 and A-2, respectively. The mode shapes for the second and third modes are plotted in Figures A-2 and A-3, respectively.

Table A-1. Closed-Closed Pipe, Natural Frequencies

i	FEM λ_i	FEM ω_i (rad/sec)	FEM f_i (Hz)	Classical Solution f_i (Hz)
1	0	0	0	0
2	3.0	3.464 c / L	0.551 c / L	0.5 c / L
3	12.0	6.928 c / L	1.103 c / L	1.0 c / L

Table A-2. Closed-Closed Pipe,
Pressure Eigenvectors with Arbitrary Scale

x / L	Mode 1	Mode 2	Mode 3
0	0.577	0.707	0.577
0.5	0.577	0.000	-0.577
1.0	0.577	-0.707	0.577

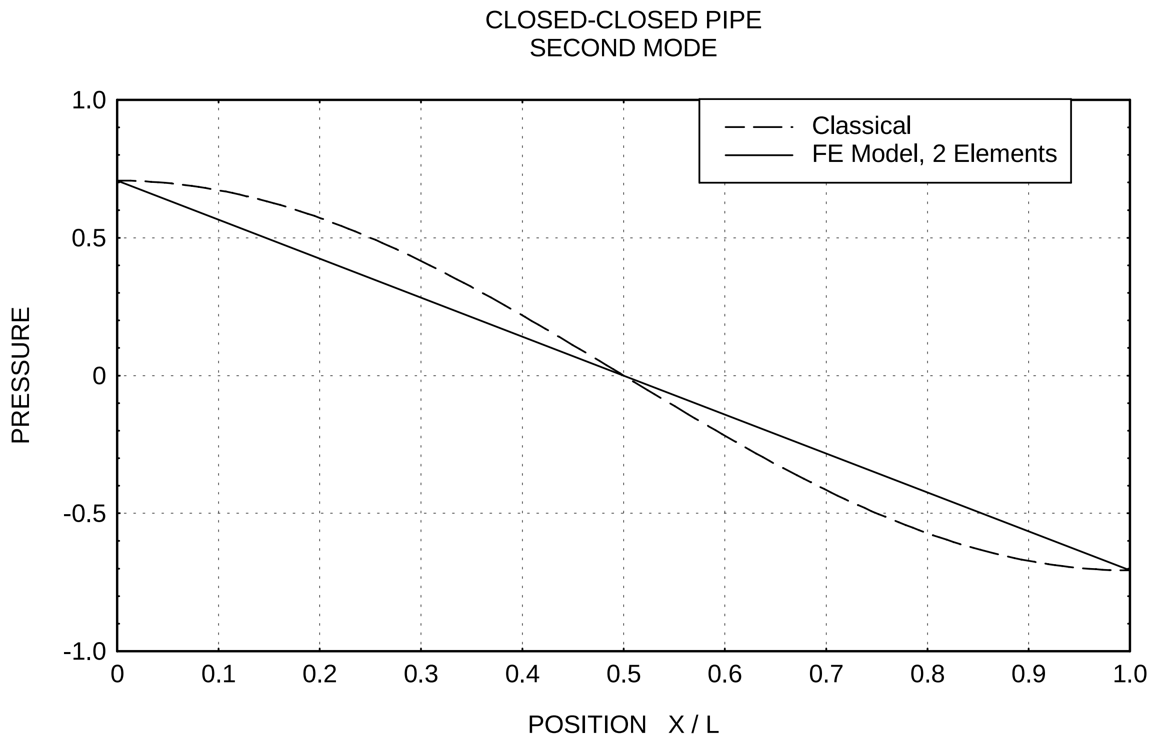


Figure A-2.

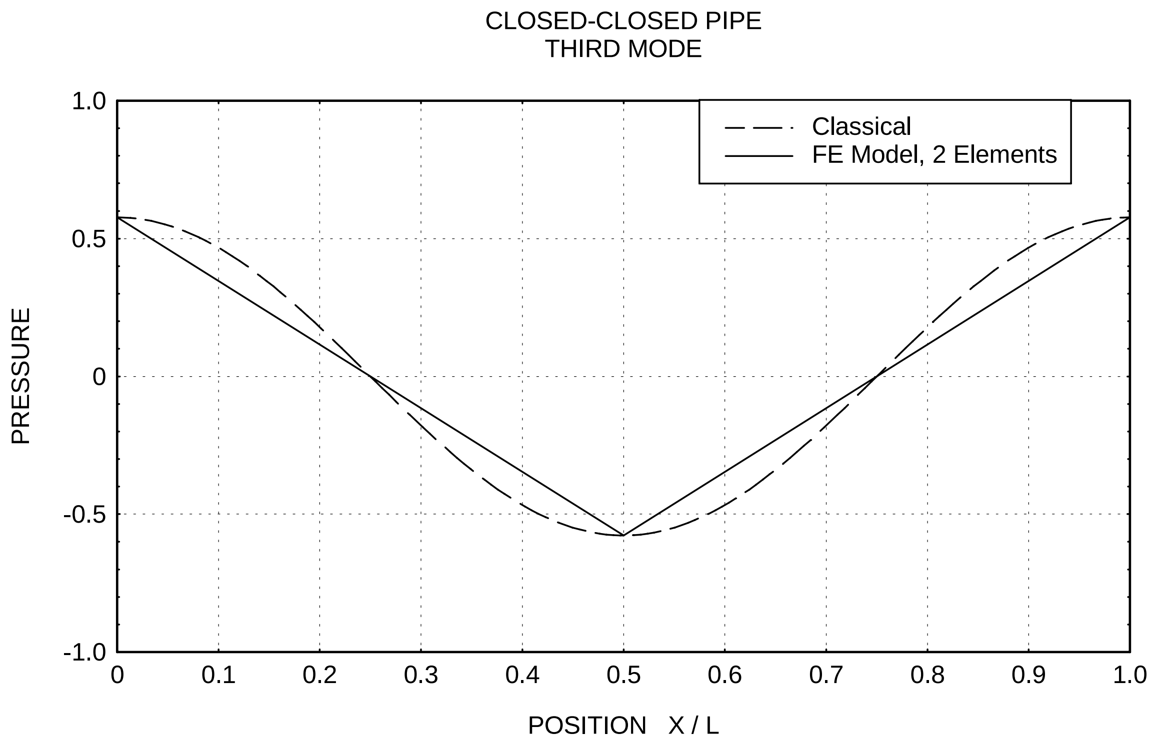


Figure A-3.

APPENDIX B

Example 1: Closed-Open Pipe, FE Model, Four Elements

The finite element model of the pipe is shown in Figure B-1. It consists of four elements and five nodes. The pipe has length L. Each element has an equal length.

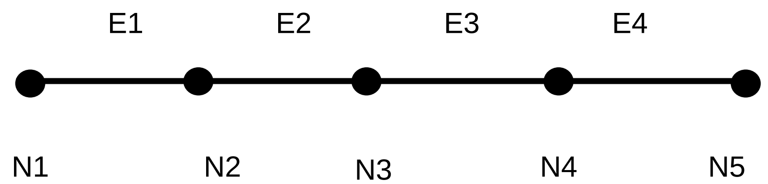


Figure B-1.

The boundary conditions are

$$\left.\frac{dP}{dx}\right|_{x=0} = 0 \qquad \text{(closed end)} \qquad \text{(B-1)}$$

$$P(L) = 0 \qquad \text{(open end)} \qquad \text{(B-2)}$$

The generalized eigenvalue problem with global mass and stiffness matrices is assemble in the same manner as the example in Appendix A. The open boundary condition must be considered, however.

Application of the $P(L) = 0$ boundary condition causes each entry in the last column and last row of each matrix to equal zero. The last column and last row are thus removed from the problem. The resulting eigenvalue problem is

$$\det\left\{\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} - \left(\frac{\lambda}{6}\right)\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}\right\} = 0 \qquad \text{(B-4)}$$

Recall

$$\lambda = \left(\frac{h\ \omega}{c} \right)^2 \tag{B-5}$$

$$h = L / 4 \tag{B-6}$$

The eigenvalues can be obtained via Matlab as follows.

```
k =

     1    -1     0     0
    -1     2    -1     0
     0    -1     2    -1
     0     0    -1     2


m =

    0.3333    0.1667         0         0
    0.1667    0.6667    0.1667         0
         0    0.1667    0.6667    0.1667
         0         0    0.1667    0.6667

>> [ModeShapes,Eigenvalues]=eig(k,m);
>> Eigenvalues

Eigenvalues =

    0.1562         0         0         0
         0    1.5545         0         0
         0         0    5.1295         0
         0         0         0   10.7268
```

The resulting natural frequencies are given in Table B-1. The first and second mode shapes are plotted in Figures B-2 and B-3, respectively. Each mode shape has an arbitrary scale factor.

Table B-1. Closed-Open Pipe, Natural Frequencies				
i	FEM λ_i	FEM ω_i (rad/sec)	FEM f_i (Hz)	Classical Solution f_i (Hz)
1	0.1562	1.581 c / L	0.25 c / L	0.25 c / L
2	1.5545	4.987 c / L	0.79 c / L	0.75 c / L
3	5.1295	9.060 c / L	1.44 c / L	1.25 c / L
4	10.7268	13.101 c / L	2.09 c / L	1.75 c / L

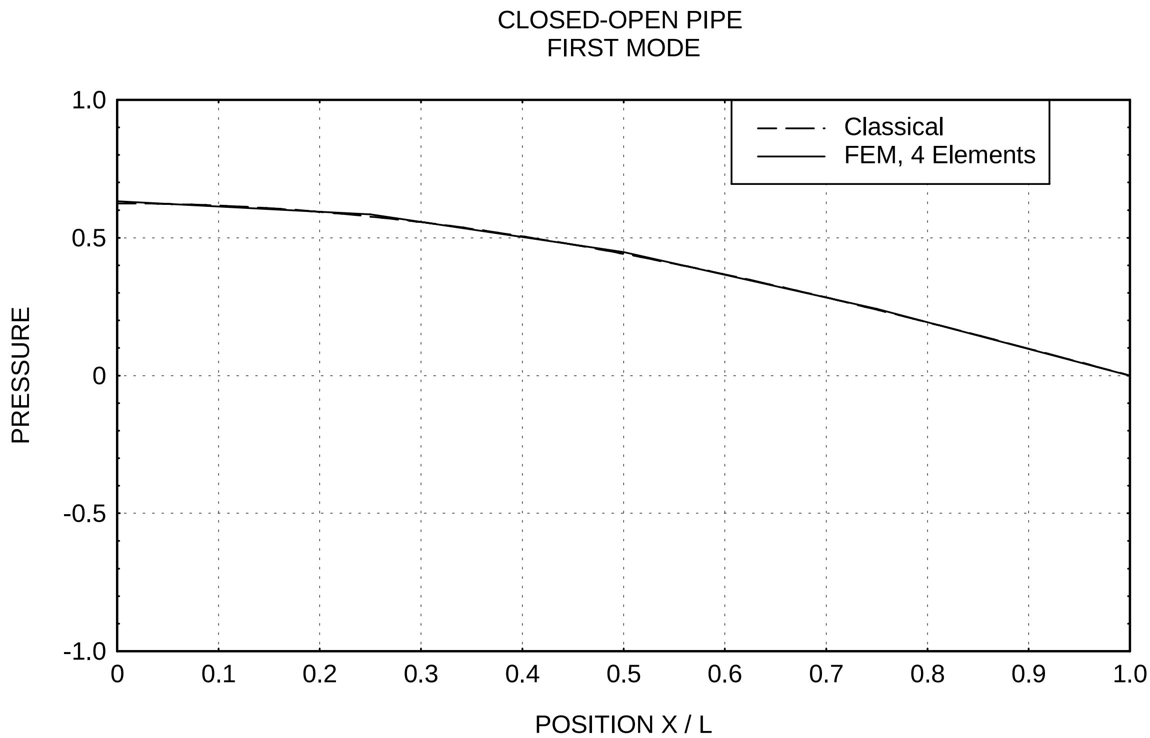


Figure B-2.

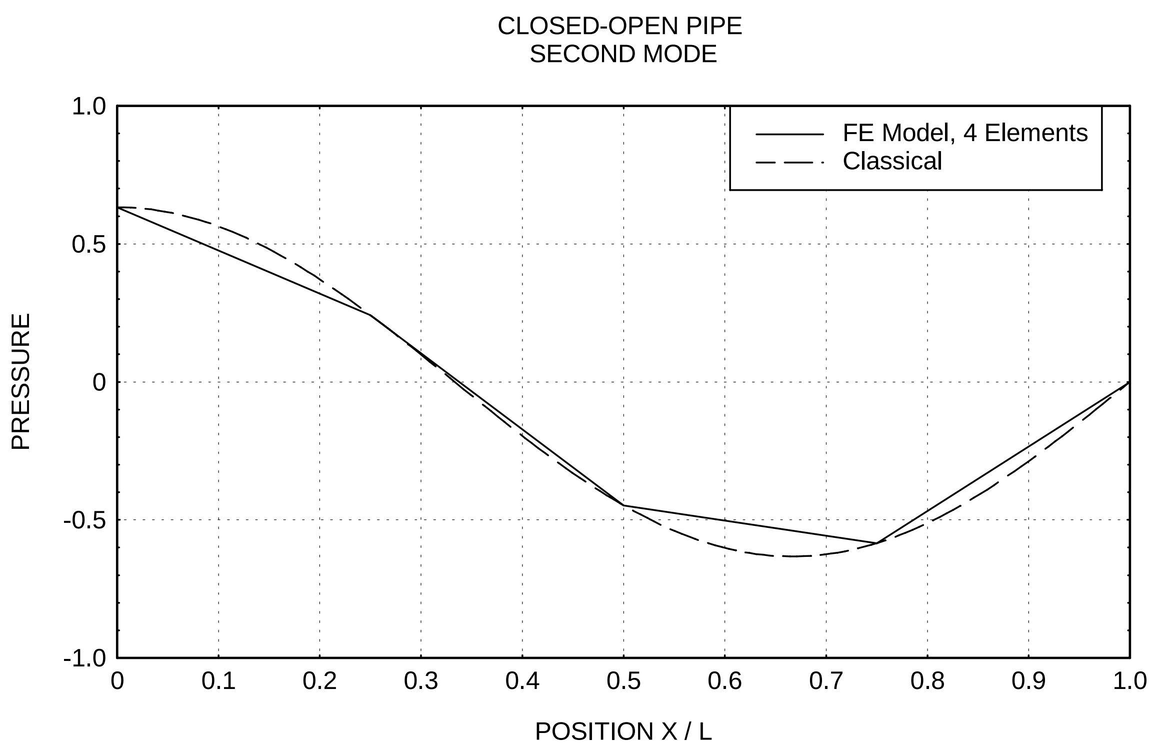


Figure B-3.

APPENDIX C

Virtual Work Principle for a One-Dimensional Acoustic Element

If a general structure in dynamic equilibrium is subjected to a system of small virtual displacements within a compatible sate of deformation, the virtual work of external actions is equal to the virtual strain energy of internal stresses.

$$\delta U_e = \delta W_e \tag{C-1}$$

where

δU_e is the virtual strain energy of internal stresses

δW_e is the virtual work of external actions on the element

Assume a vector of small virtual displacements

$$\delta \bar{q} = \{\delta \bar{q}_i\}, \quad i = 1, 2, \ldots, n \tag{C-2}$$

The resulting virtual generic displacement becomes

$$\delta u = \underline{L}\ \delta \bar{q} \tag{C-3}$$

where

$\underline{L}$ is a matrix of shape functions

The time varying strain $\varepsilon(t)$ is

$$\varepsilon(t) = \frac{d}{dx}\, u(t) \tag{C-4}$$

The displacement function is

$$u(t) = \underline{L}\, \bar{q} \tag{C-5}$$

By substitution

$$\varepsilon(t) = \frac{d}{dx} \underline{L} \ \overline{q} \tag{C-6}$$

$$\delta\varepsilon(t) = \frac{d}{dx} \underline{L} \ \delta\overline{q} \tag{C-7}$$

Thus

$$\varepsilon(t) = \underline{L'} \ \overline{q} \tag{C-8}$$

$$\delta\varepsilon(t) = \underline{L'} \ \delta\overline{q} \tag{C-9}$$

The internal virtual strain energy is

$$\delta U_e = \int_a^b \delta\varepsilon \ \sigma \ dx \tag{C-10}$$

The integration is performed over the length of the element.

Now let

$p(t)$ = applied nodal pressure

ρ = mass density

The external virtual work is

$$\delta W_e = \delta\overline{q}^T \ \overline{p}(t) - \int_a^b \delta u \ \rho \ddot{u} \ dx \tag{C-11}$$

Equate the internal virtual strain energy with external virtual work.

$$\int_a^b \delta\varepsilon \ \sigma \ dx = \delta\overline{q}^T \ \overline{p}(t) - \int_a^b \delta u \ \rho \ddot{u} \ dx \tag{C-12}$$

$$\sigma = E \varepsilon \tag{C-13}$$

$$\int_a^b \delta\varepsilon\, E\,\varepsilon\, dx \;=\; \delta\bar{q}^T\, \bar{p}(t) \;-\; \int_a^b \delta u\, \rho\, \ddot{u}\, dx \qquad \text{(C-14)}$$

$$\delta\varepsilon(t) = \underline{L}'\ \delta\bar{q} \qquad \text{(C-15)}$$

$$\varepsilon(t) = \underline{L}'\,\bar{q} \qquad \text{(C-16)}$$

$$\int_a^b \underline{L}'\ \delta\bar{q}\ E\,\underline{L}'\,\bar{q}\, dx \;=\; \delta\bar{q}^T\, \bar{p}(t) \;-\; \int_a^b \delta u\, \rho\, \ddot{u}\, dx \qquad \text{(C-17)}$$

$$\delta\bar{q}^T \left\{ \int_a^b \underline{L}'^T\ E\,\underline{L}\ dx \right\} \bar{q} \;=\; \delta\bar{q}^T\, \bar{p}(t) \;-\; \int_a^b \delta u\, \rho\, \ddot{u}\, dx \qquad \text{(C-18)}$$

$$u(t) = \underline{L}\,\bar{q} \qquad \text{(C-19)}$$

$$\delta u = \underline{L}\ \delta\bar{q} \qquad \text{(C-20)}$$

$$\delta\bar{q}^T \left\{ \int_a^b \underline{L}'^T\ E\,\underline{L}'\ dx \right\} \bar{q} \;=\; \delta\bar{q}^T\, \bar{p}(t) \;-\; \delta\bar{q}^T \left\{ \int_a^b \underline{L}^T\ \rho\,\underline{L}\ dx \right\} \ddot{\bar{q}} \qquad \text{(C-21)}$$

$$\left\{ \int_a^b \underline{L}'^T\ E\,\underline{L}'\ dx \right\} \bar{q} \;=\; \bar{p}(t) \;-\; \left\{ \int_a^b \underline{L}^T\ \rho\,\underline{L}\ dx \right\} \ddot{\bar{q}} \qquad \text{(C-22)}$$

$$\left\{ \int_a^b \underline{L}^T\ \rho\,\underline{L}\ dx \right\} \ddot{\bar{q}} \;+\; \left\{ \int_a^b \underline{L}'^T\ E\,\underline{L}'\ dx \right\} \bar{q} \;=\; \bar{p}(t) \qquad \text{(C-23)}$$

E and ρ are constant for air

$$\left\{ \rho \int_a^b \underline{L}^T\ \underline{L}\ dx \right\} \ddot{\bar{q}} \;+\; \left\{ E \int_a^b \underline{L}'^T\ \underline{L}' dx \right\} \bar{q} \;=\; \bar{p}(t) \qquad \text{(C-24)}$$

$$\rho\,\underline{M}\,\ddot{\bar{q}} \;+\; E\,\underline{K}\,\bar{q} \;=\; \bar{p}(t) \qquad \text{(C-25)}$$

The mass matrix is

$$\underline{M} = \int_a^b \underline{L}^T \ \underline{L} \ dx \qquad \text{(C-26)}$$

The stiffness matrix is

$$\underline{K} = \int_a^b \underline{L}'^T \, \underline{L}' \, dx \qquad \text{(C-27)}$$