ACOUSTIC NATURAL FREQUENCIES OF A RECTANGULAR ROOM
Revision B

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Introduction

Rooms typically contain tens of thousands of acoustics modes within the frequency domain of human hearing. Each of these modes represents a standing wave at a natural frequency. Note that these waves are longitudinal pressure waves.

The purpose of this tutorial is to derive the frequency equation for the acoustic modes. The derivation is based on References 1 and 2.

Diagram

A room diagram is given in Figure 1.

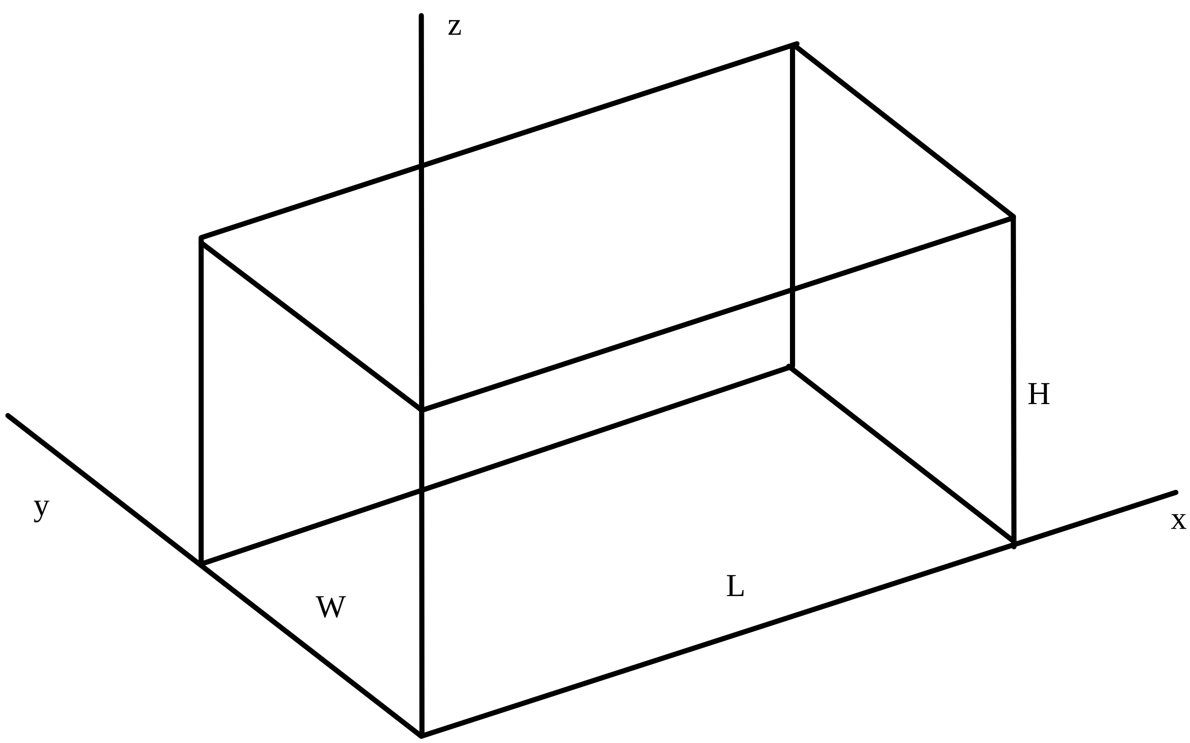

Figure 1. Rectangular Room

The room has length L, width W, and height H. These dimensions are measured with respect to the x, y, and z-axes.

Natural Frequencies

The natural frequencies are calculated from the wave equation. A complete derivation is given in Appendix A. The result for the natural frequencies f_n is given in equation (1).

$$f_n = \frac{c}{2}\sqrt{\left(\frac{p}{L}\right)^2 + \left(\frac{q}{W}\right)^2 + \left(\frac{r}{H}\right)^2} \qquad (1)$$

where

c is the speed of sound,

p, q, and r are integers, each independently equal to 0, 1, 2, 3, ...

Equation (1) assumes an enclosed, empty room with hard, reflective surfaces. The presence of carpeting, furniture, or windows could significantly alter the frequencies and mode shapes.

The speed of sound is typically 1120 ft/sec (343 meters/sec) under normal room conditions.

As an example, consider a rectangular room with length = 15 feet, width = 10 feet, height = 9 feet. The resulting natural frequencies for the first twenty modes are shown in Table 1.

Table 1. Natural Frequencies for Example

n	fn (Hz)	p	q	r
1	37.33	1	0	0
2	56.00	0	1	0
3	62.22	0	0	1
4	67.30	1	1	0
5	72.56	1	0	1
6	74.67	2	0	0
7	83.71	0	1	1
8	91.66	1	1	1
9	93.33	2	1	0
10	97.19	2	0	1
11	112.00	0	2	0
12	112.00	3	0	0
13	112.17	2	1	1
14	118.06	1	2	0
15	124.44	0	0	2
16	125.22	3	1	0
17	128.12	3	0	1
18	128.12	0	2	1
19	129.92	1	0	2
20	133.45	1	2	1

Normal Modes

The integers correspond to mode shapes. The mode types are determined by whether any of the integers are equal to zero. The types are summarized in Table 2.

Table 2. Mode Type	
Description	Integer Condition
Axial	Two integers are equal to zero
Tangential	One integer is equal to zero
Oblique	No integer is equal to zero

Note that axial modes are the most important for a sound studio design. The tangential and oblique modes tend to have a weaker effect than the axial modes.

The normal mode shape equation is

$$P(x, y, z) = m_n \cos\left(\frac{p\pi x}{L}\right)\cos\left(\frac{q\pi y}{W}\right)\cos\left(\frac{r\pi z}{H}\right) \qquad (2)$$

where

p, q, and r are each independently equal to 0, 1, 2, 3, ...

and m_n is an arbitrary scale factor.

Modal Density

The number of normal modes ΔN having frequencies in a band of width Δf centered on f is

$$\frac{\Delta N}{\Delta f} \approx \frac{4\pi V}{c^3} f^2 \qquad (3)$$

where

V is the room volume
c is the speed of sound

Equation (3) is taken from Reference 1.

The number of normal modes in a given frequency band Δf increases rapidly as the center frequency of the band increases. The number also increases as the room volume increases.

Acoustics

Dominant standing waves can be a problem in churches and auditoriums. The reverberant sound field from any source must quickly become diffuse to produce a good blend of sound throughout the room. Curved panels may be mounted on the walls to achieve this diffuse effect, as shown in Figure 2.

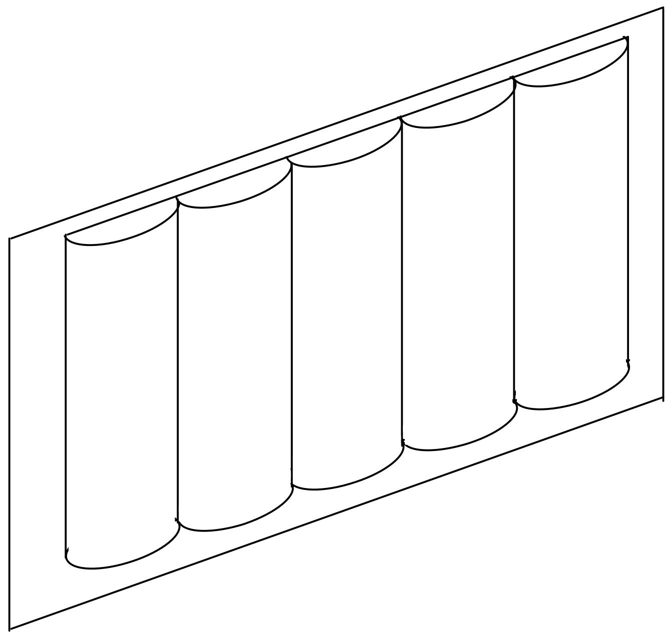

Figure 2.

Diffuse reverberation can provide a feeling of "warmth," particularly during music performances. "When satisfactory diffusion has been achieved, listeners will have the sensation of sound coming from all directions at equal levels," according to Egan, Reference 3.

Excessive reverberation, however, makes speech difficult. Thus, some compromise is usually needed in an auditorium used for both speech and music.

References

1. Lawrence Kinsler et al, Fundamentals of Acoustics, Third Edition, Wiley, New York, 1982.
2. Alton Everest, The Master Handbook of Acoustics, Tab Books, Blue Ridge Summit, PA, 1981.

3. Egan, David M., *Architectural Acoustics*, McGraw Hill, Inc., New York 1988.

APPENDIX A

The three-dimensional wave equation in rectangular coordinates for the pressure p is

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{A-1}$$

Now perform a variable separation as follows.

$$p(x,y,z,t)=P(x,y,z)T(t) \tag{A-2}$$

$$\frac{\partial^2}{\partial x^2}P(x,y,z)T(t)+\frac{\partial^2}{\partial y^2}P(x,y,z)T(t)+\frac{\partial^2}{\partial z^2}P(x,y,z)T(t)=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}P(x,y,z)T(t) \tag{A-3}$$

$$T(t)\left\{\frac{\partial^2}{\partial x^2}P(x,y,z)+\frac{\partial^2}{\partial y^2}P(x,y,z)+\frac{\partial^2}{\partial z^2}P(x,y,z)\right\}=\frac{1}{c^2}P(x,y,z)\left\{\frac{d^2}{dt^2}T(t)\right\} \tag{A-4}$$

$$\frac{c^2}{P(x,y,z)}\left\{\frac{\partial^2}{\partial x^2}P(x,y,z)+\frac{\partial^2}{\partial y^2}P(x,y,z)+\frac{\partial^2}{\partial z^2}P(x,y,z)\right\}=\frac{1}{T(t)}\left\{\frac{d^2}{dt^2}T(t)\right\} \tag{A-5}$$

Each side of equation (A-5) must equal a constant.

$$\frac{c^2}{P(x,y,z)}\left\{\frac{\partial^2}{\partial x^2}P(x,y,z)+\frac{\partial^2}{\partial y^2}P(x,y,z)+\frac{\partial^2}{\partial z^2}P(x,y,z)\right\}=\frac{1}{T(t)}\left\{\frac{d^2}{dt^2}T(t)\right\}=-\omega^2 \tag{A-6}$$

The time equation is thus

$$\frac{1}{T(t)}\left\{\frac{d^2}{dt^2}T(t)\right\} = -\omega^2 \tag{A-7}$$

The spatial equation is

$$\frac{c^2}{P(x,y,z)}\left\{\frac{\partial^2}{\partial x^2}P(x,y,z)+\frac{\partial^2}{\partial y^2}P(x,y,z)+\frac{\partial^2}{\partial z^2}P(x,y,z)\right\} = -\omega^2$$

(A-8)

$$\left\{\frac{\partial^2}{\partial x^2}P(x,y,z)+\frac{\partial^2}{\partial y^2}P(x,y,z)+\frac{\partial^2}{\partial z^2}P(x,y,z)\right\} = -\left\{\frac{\omega}{c}\right\}^2 P(x,y,z)$$

(A-9)

Now separate the spatial variable.

$$P(x,y,z)=X(x)Y(y)Z(z) \tag{A-10}$$

$$\left\{\frac{\partial^2}{\partial x^2}X(x)Y(y)Z(z)+\frac{\partial^2}{\partial y^2}X(x)Y(y)Z(z)+\frac{\partial^2}{\partial z^2}X(x)Y(y)Z(z)\right\} = -\left\{\frac{\omega}{c}\right\}^2 X(x)Y(y)Z(z)$$

(A-11)

The partial derivatives reduce to ordinary derivatives.

$$\left\{Y(y)Z(z)X''(x) + X(x)Z(z)Y''(y) + X(x)Y(y)Z''(z)\right\} = -\left\{\frac{\omega}{c}\right\}^2 X(x)Y(y)Z(z)$$

(A-12)

Divide through by X(x)Y(y)Z(z).

$$\left\{\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)}\right\} = -\left\{\frac{\omega}{c}\right\}^2 \quad \text{(A-13)}$$

Equation (A-12) is satisfied if equations (A-13) through (A-16).

$$\frac{X''(x)}{X(x)} = -k_x^{\,2} \quad \text{(A-14)}$$

$$\frac{Y''(y)}{Y(y)} = -k_y^{\,2} \quad \text{(A-15)}$$

$$\frac{Z''(z)}{Z(z)} = -k_z^{\,2} \quad \text{(A-16)}$$

$$k_x^{\,2} + k_y^{\,2} + k_z^{\,2} = \left\{\frac{\omega}{c}\right\}^2 \quad \text{(A-17)}$$

Equation (A-14) can be restated as

$$X''(x) + k_x^{\,2}\, X(x) = 0 \quad \text{(A-18)}$$

Propose a solution

$$X(x) = a\sin(k_x x) + b\cos(k_x x) \quad \text{(A-19)}$$

$$X'(x) = k_x\left\{a\cos(k_x x) - b\sin(k_x x)\right\} \quad \text{(A-20)}$$

$$X''(x) = k_x^{\,2}\left\{-a\sin(k_x x) - b\cos(k_x x)\right\} \quad \text{(A-21)}$$

By inspection, equation (A-18) is satisfied by equations (A-18) and (A-21).

The boundary conditions are such that the slope of the pressure is zero at each end. Apply these conditions to equation (A-20).

$$X'(0) = 0 \tag{A-22}$$

$$k_x a = 0 \tag{A-23}$$

$$a = 0 \tag{A-24}$$

Substitute equation (A-23) into (A-18) and (A-19).

$$X(x) = b\cos\left(k_x x\right) \tag{A-25}$$

$$X'(x) = -k_x\left\{b\sin\left(k_x x\right)\right\} \tag{A-26}$$

The second boundary condition is

$$X'(L) = 0 \tag{A-27}$$

$$-k_x\left\{b\sin\left(k_x L\right)\right\} = 0 \tag{A-28}$$

$$k_x L = p\pi, \quad p = 0,1,2,3,..... \tag{A-29}$$

$$k_x = \frac{p\pi}{L}, \quad p = 0,1,2,3,..... \tag{A-30}$$

Similarly,

$$k_y = \frac{q\pi}{W}, \quad q = 0,1,2,3,..... \tag{A-31}$$

$$k_z = \frac{r\pi}{H}, \quad r = 0,1,2,3,..... \tag{A-32}$$

Now substitute equations (A-30) through (A-32) into (A-17). A subscript n is added to the omega term to denote multiple roots.

$$\left\{\frac{\omega_n}{c}\right\}^2 = \left\{\frac{p\,\pi}{L}\right\}^2 + \left\{\frac{q\,\pi}{W}\right\}^2 + \left\{\frac{r\,\pi}{H}\right\}^2 \tag{A-33}$$

$$\left\{\frac{\omega_n}{c}\right\} = \sqrt{\left\{\frac{p\,\pi}{L}\right\}^2 + \left\{\frac{q\,\pi}{W}\right\}^2 + \left\{\frac{r\,\pi}{H}\right\}^2} \tag{A-34}$$

$$\omega_n = c\sqrt{\left\{\frac{p\,\pi}{L}\right\}^2 + \left\{\frac{q\,\pi}{W}\right\}^2 + \left\{\frac{r\,\pi}{H}\right\}^2} \tag{A-35}$$

$$2\pi f_n = c\,\pi\sqrt{\left\{\frac{p}{L}\right\}^2 + \left\{\frac{q}{W}\right\}^2 + \left\{\frac{r}{H}\right\}^2} \tag{A-36}$$

$$f_n = \frac{c}{2}\sqrt{\left(\frac{p}{L}\right)^2 + \left(\frac{q}{W}\right)^2 + \left(\frac{r}{H}\right)^2} \tag{A-37}$$

where p, q, and r are each independently equal to 0, 1, 2, 3, ...

The normal mode shapes are given by

$$P(x,y,z) = m_n \cos(k_x x)\cos(k_y y)\cos(k_z z) \tag{A-38}$$

where m_n is an arbitrary scale factor.

The equivalent mode shape equation is

$$P(x,y,z) = m_n \cos\left(\frac{p\pi x}{L}\right)\cos\left(\frac{q\pi y}{W}\right)\cos\left(\frac{r\pi z}{H}\right) \tag{A-39}$$

where p, q, and r are each independently equal to 0, 1, 2, 3, ...