ACOUSTIC OSCILLATION IN A TWO-DIMENSIONAL PRESSURE FIELD VIA THE FINITE ELEMENT METHOD WITH TRIANGULAR ELEMENTS Revision B

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Introduction

The purpose of this tutorial is to derive for a method for analyzing the acoustic pressure oscillation in a two-dimensional pressure field using the finite element method. The method is based on Reference 1.

Let p(x, y, t) represent the pressure in the field as a function of space and time.

The free, transverse vibration of the pressure field is governed by the equation:

$$c^2 \left\{ \frac{\partial^2}{\partial x^2} p(x,y,t) + \frac{\partial^2}{\partial y^2} p(x,y,t) \right\} = \frac{\partial^2}{\partial t^2} p(x,y,t) \tag{1}$$

Equation (1) is independent of the boundary conditions, which are applied as constraint equations.

Assume that the solution of equation (1) is separable in time and space.

$$p(x,y,t) = P(x,y)\,f(t) \tag{2}$$

$$c^2 \left\{ \frac{\partial^2}{\partial x^2} P(x,y) f(t) + \frac{\partial^2}{\partial y^2} P(x,y) f(t) \right\} = \frac{\partial^2}{\partial t^2} P(x,y) f(t) \tag{3}$$

$$c^2\, f(t) \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] P(x,y) = P(x,y) \frac{\partial^2 f(t)}{\partial t^2} \tag{4}$$

The equation may be restated as

$$c^2 \frac{1}{P(x,y)} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] P(x,y) = \frac{1}{f(t)} \frac{d^2 f(t)}{d t^2} \tag{5}$$

The left-hand side of equation (5) depends on space only. The right hand side depends on time only. Both space and time are independent variables. Thus equation (5) only has a solution if both sides are constant. Let $-\omega^2$ be the constant.

$$c^2 \frac{1}{P(x,y)}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]P(x,y) \quad = \quad \frac{1}{f(t)}\frac{d^2 f(t)}{dt^2} \quad = \quad -\omega^2 \tag{6}$$

Equation (6) yields two independent equations.

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]P(x,y) + \frac{\omega^2}{c^2} P(x,y) = 0 \tag{7}$$

$$\frac{d^2}{dt^2} f(t) + \omega^2 f(t) = 0 \tag{8}$$

Equation (7) is a homogeneous, second order, partial differential equation.

The weighted residual method is applied to equation (7). This method is suitable for boundary value problems. An alternative method would be the energy method.

There are numerous techniques for applying the weighted residual method. Specifically, the Galerkin approach is used in this tutorial.

The differential equation (7) is multiplied by a test function $\phi(x,y)$. Note that the test function $\phi(x,y)$ must satisfy the homogeneous essential boundary conditions. The essential boundary conditions are the prescribed values of p and its first derivative.

The test function is not required to satisfy the differential equation, however.

The product of the test function and the differential equation is integrated over the area domain. The integral is set equation to zero.

$$\int_A \phi(x,y) \left\{\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]P(x,y) + \frac{\omega^2}{c^2} P(x,y)\right\} dA = 0 \tag{9}$$

The test function $\phi(x,y)$ can be regarded as a virtual pressure. The differential equation in the brackets represents an internal force. This term is also regarded as the residual.

Thus, the integral represents virtual work, which should vanish at the equilibrium condition.

Use the product rule for differentiation

$$\frac{\partial}{\partial x}\left(\phi\frac{\partial P}{\partial x}\right) = \frac{\partial \phi}{\partial x}\frac{\partial P}{\partial x} + \phi\frac{\partial^2 P}{\partial^2 x} \tag{10}$$

$$\phi\frac{\partial^2 P}{\partial^2 x} = \frac{\partial}{\partial x}\left(\phi\frac{\partial P}{\partial x}\right) - \frac{\partial \phi}{\partial x}\frac{\partial P}{\partial x} \tag{11}$$

Similarly

$$\phi\frac{\partial^2 P}{\partial^2 y} = \frac{\partial}{\partial y}\left(\phi\frac{\partial P}{\partial y}\right) - \frac{\partial \phi}{\partial y}\frac{\partial P}{\partial y} \tag{12}$$

$$\int_A \left[\frac{\partial}{\partial x}\left(\phi\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(\phi\frac{\partial P}{\partial y}\right)\right]dA - \int_A \left\{\frac{\partial \phi}{\partial x}\frac{\partial P}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial P}{\partial y}\right\}dA$$
$$+\left(\frac{\omega^2}{c^2}\right)\int_A \phi(x,y)\, P(x,y)dA = 0 \tag{13}$$

Green’s theorem

$$\int_A \left[\frac{\partial}{\partial x}\left(\phi\frac{\partial P}{\partial x}\right)\right]dA = \int_\Gamma \left[\phi\frac{\partial P}{\partial x}\right]\cos\theta \; d\Gamma \tag{14}$$

$$\int_A \left[\frac{\partial}{\partial y}\left(\phi\frac{\partial P}{\partial y}\right)\right]dA = \int_\Gamma \left[\phi\frac{\partial P}{\partial y}\right]\sin\theta \; d\Gamma \tag{15}$$

where

Γ = element boundary

θ = angle relative to the normal vector that passes the through the element centroid

$$\int_\Gamma \left\{ \left[\phi \frac{\partial P}{\partial x} \right] \cos\theta + \left[\phi \frac{\partial P}{\partial y} \right] \sin\theta \right\} d\Gamma$$

$$- \int_A \left\{ \frac{\partial \phi}{\partial x} \frac{\partial P}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial P}{\partial y} \right\} dA + \left(\frac{\omega^2}{c^2} \right) \int_A \phi(x,y)\, P(x,y) dA = 0$$

(16)

The essence of the Galerkin method is that the test function is chosen as

$$\phi(x,y) = P(x,y) \tag{17}$$

Thus

$$\int_\Gamma P \left\{ \left[\frac{\partial P}{\partial x} \right] \cos\theta + \left[\frac{\partial P}{\partial y} \right] \sin\theta \right\} d\Gamma$$

$$- \int_A \left\{ \left(\frac{\partial P}{\partial x} \right)\left(\frac{\partial P}{\partial x} \right) + \left(\frac{\partial P}{\partial y} \right)\left(\frac{\partial P}{\partial y} \right) \right\} dA + \left(\frac{\omega^2}{c^2} \right) \int_A [P(x,y)]^2 \, dA = 0$$

(18)

The boundary condition for an open boundary is that

$$P = 0 \tag{19}$$

The boundary condition for a closed boundary is that

$$\frac{\partial P}{\partial n} = 0 \tag{20}$$

In either case,

$$P \left\{ \left[\frac{\partial P}{\partial x} \right] \cos\theta + \left[\frac{\partial P}{\partial y} \right] \sin\theta \right\} = 0 \tag{21}$$

Thus

$$-\int_A \left\{ \left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial P}{\partial y}\right) \right\} dA + \left(\frac{\omega^2}{c^2}\right) \int_A \left[P(x,y)\right]^2 \, dA = 0 \tag{22}$$

Develop interpolation functions for a triangular element as shown in Figure 1.

The interpolation polynomial is

$$\Psi(x,y) = \alpha_1 + \alpha_2 \, x + \alpha_3 \, y \tag{23}$$

The nodal conditions are

$$\Psi(X_i, Y_i) = \Psi_i \tag{24}$$

$$\Psi(X_j, Y_j) = \Psi_j \tag{25}$$

$$\Psi(X_k, Y_k) = \Psi_k \tag{26}$$

By substitution,

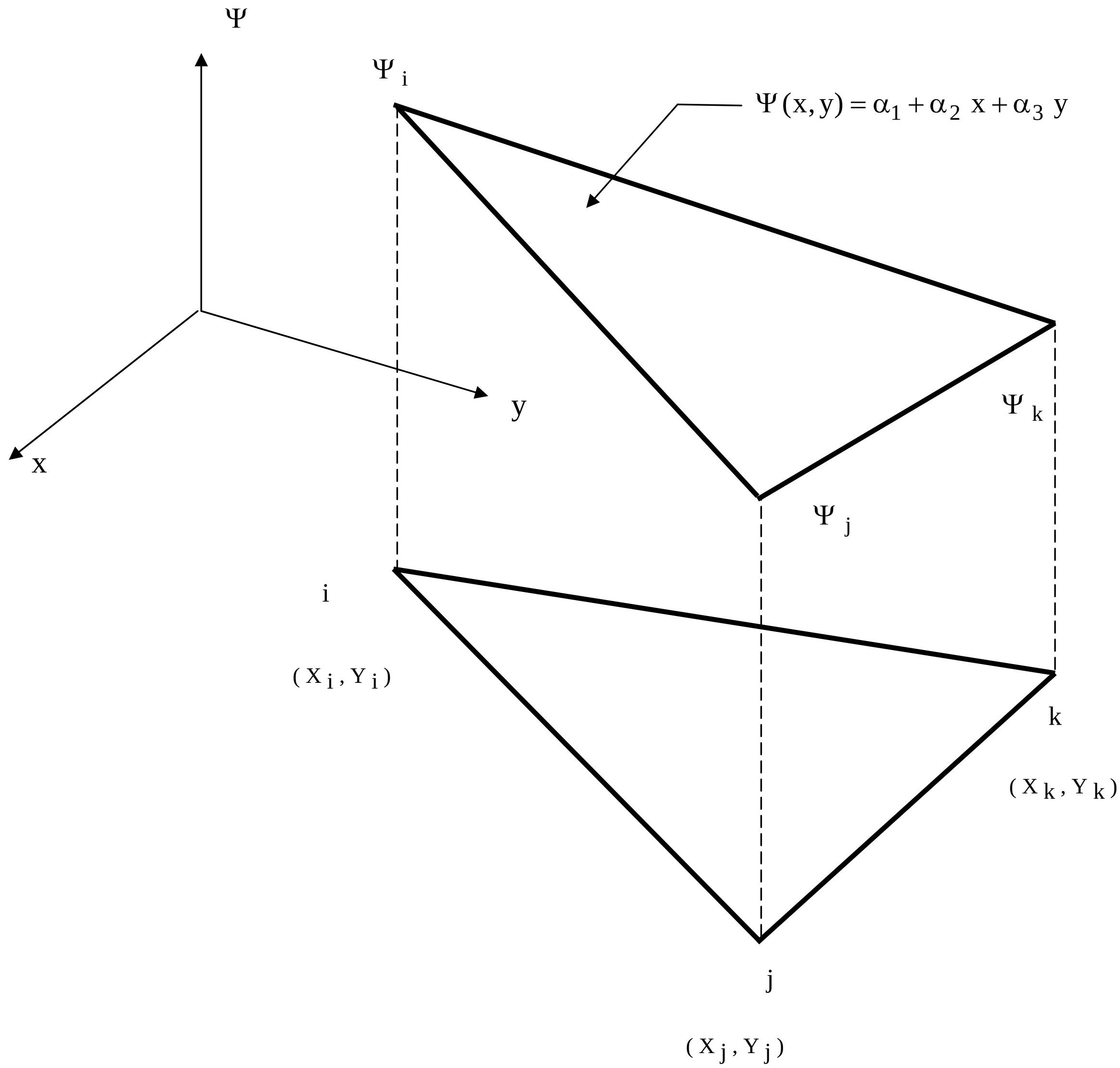
$$\Psi_i = \alpha_1 + \alpha_2 \, X_i + \alpha_3 \, Y_i \tag{27}$$

$$\Psi_j = \alpha_1 + \alpha_2 \, X_j + \alpha_3 \, Y_j \tag{28}$$

$$\Psi_k = \alpha_1 + \alpha_2 \, X_k + \alpha_3 \, Y_k \tag{29}$$

The coefficients can be calculated via the following equation in matrix form.

$$\begin{bmatrix} 1 & X_i & Y_i \\ 1 & X_j & Y_j \\ 1 & X_k & Y_k \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \Psi_i \\ \Psi_j \\ \Psi_k \end{bmatrix} \tag{30}$$



Ψ
Ψ_i
$\Psi(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y$
y
x
Ψ_k
Ψ_j
i
(X_i, Y_i)
k
(X_k, Y_k)
j
(X_j, Y_j)

Figure 1.

$$\det \begin{bmatrix} 1 & X_i & Y_i \\ 1 & X_j & Y_j \\ 1 & X_k & Y_k \end{bmatrix} = \left(X_j Y_k - X_k Y_j\right) - \left(X_i Y_k - X_k Y_i\right) + \left(X_i Y_j - X_j Y_i\right)$$

(31)

$$2A = \left(X_j Y_k - X_k Y_j\right) - \left(X_i Y_k - X_k Y_i\right) + \left(X_i Y_j - X_j Y_i\right)$$

(32)

$$\det \begin{bmatrix} 1 & X_i & Y_i \\ 1 & X_j & Y_j \\ 1 & X_k & Y_k \end{bmatrix} = 2A$$

(33)

where A is the area of the triangle.

$$\alpha_1 = \left(\frac{1}{2A}\right) \det \begin{bmatrix} \Psi_i & X_i & Y_i \\ \Psi_j & X_j & Y_j \\ \Psi_k & X_k & Y_k \end{bmatrix}$$

(34)

$$\alpha_1 =$$

$$\left(\frac{1}{2A}\right)\left[\Psi_i \left(X_j Y_k - X_k Y_j\right) + \Psi_j \left(X_k Y_i - X_i Y_k\right) + \Psi_k \left(X_i Y_j - X_j Y_i\right)\right]$$

(35)

$$\alpha_2 = \left(\frac{1}{2A}\right) \det \begin{bmatrix} 1 & \Psi_i & Y_i \\ 1 & \Psi_j & Y_j \\ 1 & \Psi_k & Y_k \end{bmatrix} \tag{36}$$

$$\alpha_2 = \left(\frac{1}{2A}\right) \left[\left(\Psi_j Y_k - \Psi_k Y_j\right) - \left(\Psi_i Y_k - \Psi_k Y_i\right) + \left(\Psi_i Y_j - \Psi_j Y_i\right) \right] \tag{37}$$

$$\alpha_2 = \left(\frac{1}{2A}\right) \left[\Psi_i\left(Y_j - Y_k\right) + \Psi_j\left(Y_k - Y_i\right) + \Psi_k\left(Y_i - Y_j\right) \right] \tag{38}$$

$$\alpha_3 = \left(\frac{1}{2A}\right) \det \begin{bmatrix} 1 & X_i & \Psi_i \\ 1 & X_j & \Psi_j \\ 1 & X_k & \Psi_k \end{bmatrix} \tag{39}$$

$$\alpha_3 = \left(\frac{1}{2A}\right) \left[\left(X_j \Psi_k - X_k \Psi_j\right) + \left(X_k \Psi_i - X_i \Psi_k\right) + \left(X_i \Psi_j - X_j \Psi_i\right) \right] \tag{40}$$

$$\alpha_3 = \left(\frac{1}{2A}\right) \left[\Psi_i\left(X_k - X_j\right) + \Psi_j\left(X_i - X_k\right) + \Psi_k\left(X_j - X_i\right) \right] \tag{41}$$

$\Psi(x, y)$

$$= \left\{ \left(\frac{1}{2A} \right) \left[\Psi_i \left(X_j Y_k - X_k Y_j \right) + \Psi_j \left(X_k Y_i - X_i Y_k \right) + \Psi_k \left(X_i Y_j - X_j Y_i \right) \right] \right\}$$

$$+ x \left\{ \left(\frac{1}{2A} \right) \left[\Psi_i \left(Y_j - Y_k \right) + \Psi_j \left(Y_k - Y_i \right) + \Psi_k \left(Y_i - Y_j \right) \right] \right\}$$

$$+ y \left\{ \left(\frac{1}{2A} \right) \left[\Psi_i \left(X_k - X_j \right) + \Psi_j \left(X_i - X_k \right) + \Psi_k \left(X_j - X_i \right) \right] \right\}$$

(42)

$\Psi(x, y) =$

$$\Psi_i \left\{ \left(\frac{1}{2A} \right) \left[\left(X_j Y_k - X_k Y_j \right) + x \left(Y_j - Y_k \right) + y \left(X_k - X_j \right) \right] \right\}$$
$$+ \Psi_j \left\{ \left(\frac{1}{2A} \right) \left[\left(X_k Y_i - X_i Y_k \right) + x \left(Y_k - Y_i \right) + y \left(X_i - X_k \right) \right] \right\}$$
$$+ \Psi_k \left\{ \left(\frac{1}{2A} \right) \left[\left(X_i Y_j - X_j Y_i \right) + x \left(X_i Y_j - X_j Y_i \right) + y \left(X_j - X_i \right) \right] \right\}$$

(43)

$$\Psi(x, y) = N_i \Psi_i + N_j \Psi_j + N_k \Psi_k \tag{44}$$

where

$$N_i = \left\{ \left(\frac{1}{2A} \right) \left[\left(X_j Y_k - X_k Y_j \right) + x \left(Y_j - Y_k \right) + y \left(X_k - X_j \right) \right] \right\} \tag{45a}$$

$$N_j = \left\{ \left(\frac{1}{2A} \right) \left[\left(X_k Y_i - X_i Y_k \right) + x \left(Y_k - Y_i \right) + y \left(X_i - X_k \right) \right] \right\} \tag{45b}$$

$$N_k = \left\{ \left(\frac{1}{2A} \right) \left[\left(X_i Y_j - X_j Y_i \right) + x \left(X_i Y_j - X_j Y_i \right) + y \left(X_j - X_i \right) \right] \right\} \tag{45c}$$

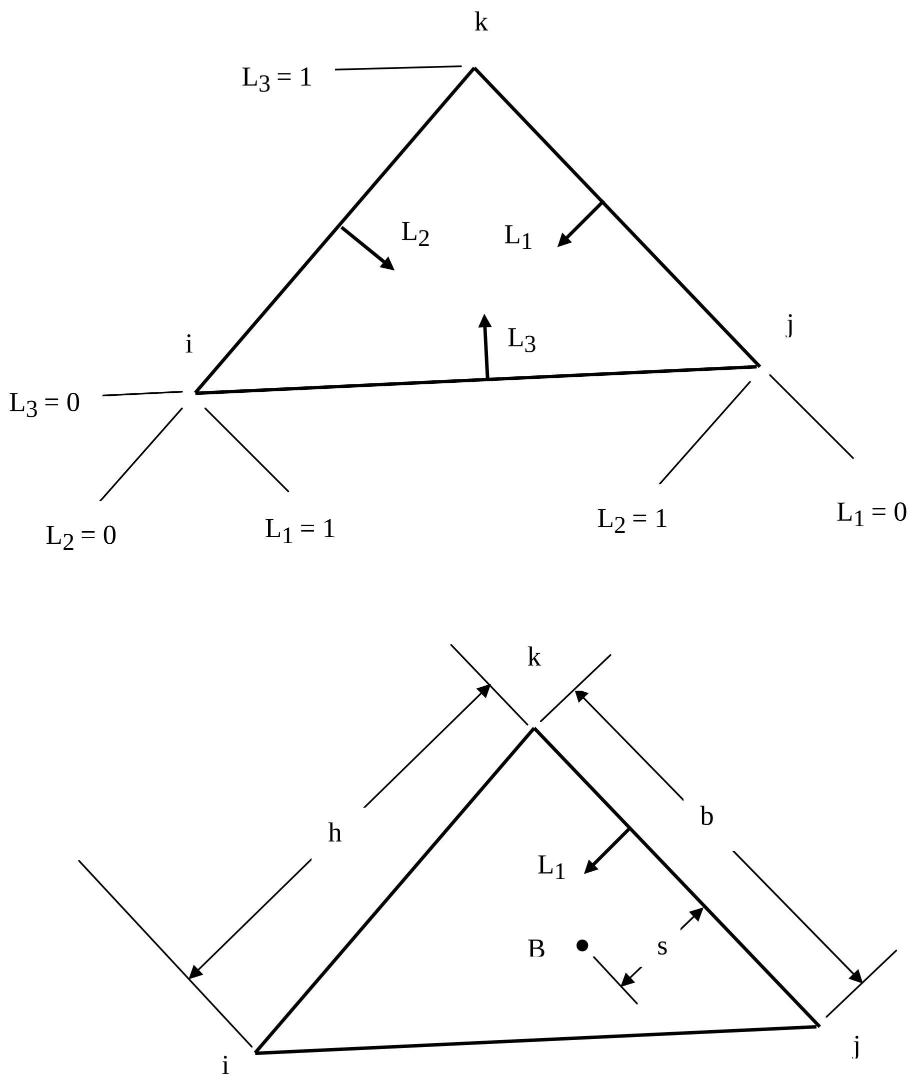


Figure 2.

A natural coordinate system is formed by defining three length ratios L_1, L_2, and L_3 , as shown in Figure 2.

Furthermore, each coordinate is the ratio of a perpendicular distance from one side s to the altitude h of that same side, as shown in Figure 2.   Furthermore, each coordinate varies from zero to 1.

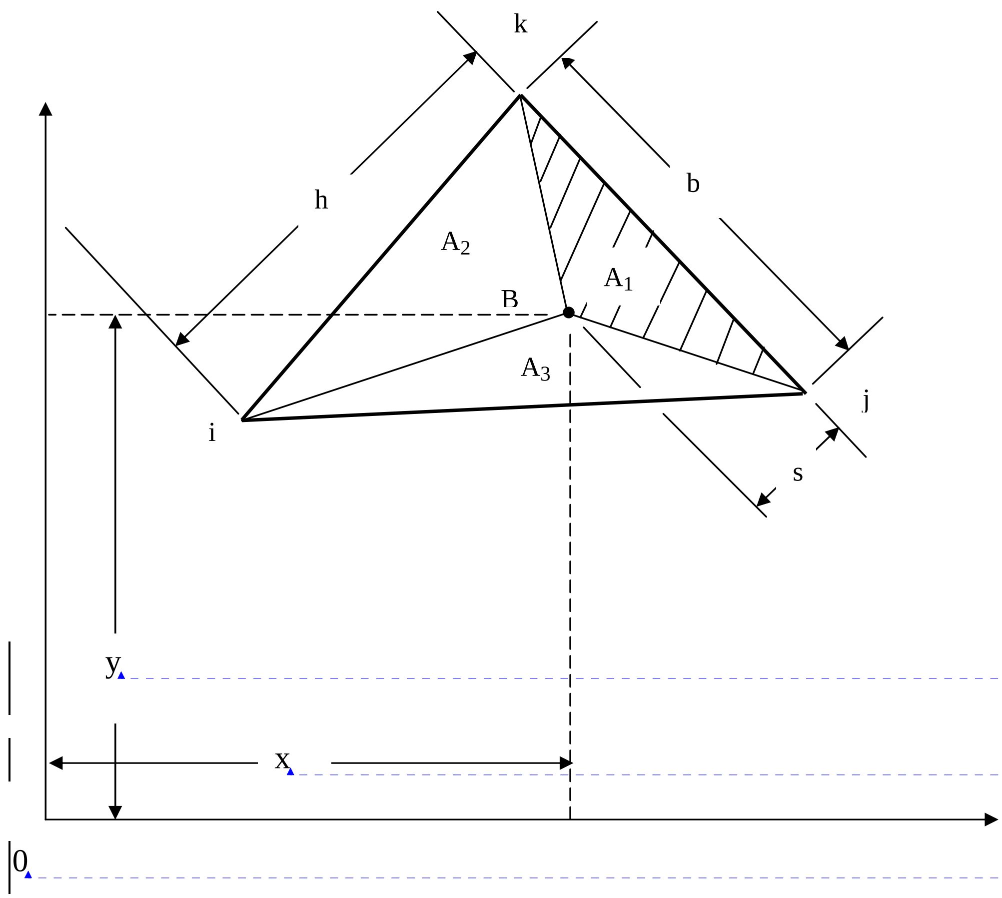


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Figure 3.

Note that

$$L_1 \;=\; s / h \tag{46}$$

The area of the complete triangle is A and is given by

$$A = b\,h / 2 \tag{47}$$

The area of triangle A_1 is

$$A_1 = b\ s / 2 \tag{48}$$

The following ratio can be formed.

$$\frac{A_1}{A} = \frac{s}{h} \tag{49}$$

$$\frac{A_1}{A} = L_1 \tag{50}$$

Similarly,

$$\frac{A_2}{A} = L_2 \tag{51}$$

$$\frac{A_3}{A} = L_3 \tag{52}$$

Note that

$$A_1 + A_2 + A_3 = A \tag{53}$$

$$\frac{1}{A}\{A_1 + A_2 + A_3\} = 1 \tag{54}$$

Thus

$$L_1 + L_2 + L_3 = 1 \tag{55}$$

The three coordinates are not independent. Thus the location of a point can be specified using two of the coordinates.

$$L_1 = \frac{2\,A_1}{2\,A} \tag{56}$$

$$2A_1 = 2\,A\,L_1 \tag{57}$$

$$2A_1 = 2\left(\frac{b\,h}{2}\right)\left(\frac{s}{h}\right) \tag{58}$$

$$2A_1 = b\,s \tag{59}$$

Recall

$$2A = \left(X_j Y_k - X_k Y_j\right) - \left(X_i Y_k - X_k Y_i\right) + \left(X_i Y_j - X_j Y_i\right) \tag{60}$$

Thus

$$2A_1 = \left(X_j Y_k - X_k Y_j\right) - \left(x\,Y_k - X_k y\right) + \left(x\,Y_j - X_j\,y\right) \tag{61}$$

$$2A_1 = \left(X_j Y_k - X_k Y_j\right) + x\left(Y_j - Y_k\right) + y\left(X_k - X_j\right) \tag{62}$$

Thus

$$L_1 = \frac{1}{2A}\left[\left(X_j Y_k - X_k Y_j\right) + x\left(Y_j - Y_k\right) + y\left(X_k - X_j\right)\right] \tag{63}$$

Recall

$$N_i = \left\{\left(\frac{1}{2A}\right)\left[\left(X_j Y_k - X_k Y_j\right) + x\left(Y_j - Y_k\right) + y\left(X_k - X_j\right)\right]\right\} \tag{64}$$

Thus

$$L_1 = N_1 \tag{65}$$

Similarly,

$$L_2 = N_2 \tag{66}$$

$$L_3 = N_3 \tag{67}$$

The area coordinates are identical to the shape functions.

The shape functions may be evaluated using a formula from Eisenberg and Malvern.

$$\int_A L_1^a\, L_2^b\, L_3^c\, dA \quad = \quad \frac{a!\,b!\,c!}{(a+b+c+2)!}\,2A \tag{68}$$

Recall

$$-\int_A \left\{ \left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial P}{\partial y}\right) \right\} dA \;+ \left(\frac{\omega^2}{c^2}\right) \int_A \left[P(x,y)\right]^2\, dA = 0 \tag{69}$$

Let

$$P(x,y) = \underline{L}^{T}\ \bar{p} \tag{70}$$

$$-\int_A \left\{ \left(\frac{\partial}{\partial x}\left[\underline{L}^{T}\,\bar{p}\right]\right)\left(\frac{\partial}{\partial x}\left[\underline{L}^{T}\,\bar{p}\right]\right) + \left(\frac{\partial}{\partial y}\left[\underline{L}^{T}\,\bar{p}\right]\right)\left(\frac{\partial}{\partial y}\left[\underline{L}^{T}\,\bar{p}\right]\right) \right\} dA$$

$$+\left(\frac{\omega^2}{c^2}\right) \int_A \left[\underline{L}^{T}\ \bar{p}\right]\left[\underline{L}^{T}\ \bar{p}\right] dA = 0 \tag{71}$$

$$-\ \bar{p}^T \int_A \left\{ \left(\frac{\partial}{\partial x}\left[\underline{L}\right]\right)\left(\frac{\partial}{\partial x}\left[\underline{L}^{T}\right]\right) + \left(\frac{\partial}{\partial y}\left[\underline{L}\right]\right)\left(\frac{\partial}{\partial y}\left[\underline{L}^{T}\right]\right) \right\} dA\ \bar{p}$$

$$+\left(\frac{\omega^2}{c^2}\right) \bar{p}^T \int_A \left[\underline{L}\ \underline{L}^{T}\right] dA\ \bar{p} = 0 \tag{72}$$

$$- \int_A \left\{ \left(\frac{\partial}{\partial x}\left[\underline{L}\right]\right)\left(\frac{\partial}{\partial x}\left[\underline{L}^T\right]\right) + \left(\frac{\partial}{\partial y}\left[\underline{L}\right]\right)\left(\frac{\partial}{\partial y}\left[\underline{L}^T\right]\right) \right\} dA$$

$$+ \left(\frac{\omega^2}{c^2}\right) \int_A \left[\underline{L}\ \underline{L}^T\right] dA = 0$$

(73)

$$\det\left\{ \underline{K} + \left(\frac{\omega^2}{c^2}\right) \underline{M} \right\} = 0 \qquad (74)$$

where

$$\underline{K} = \int_A \left\{ \left(\frac{\partial}{\partial x}\left[\underline{L}\right]\right)\left(\frac{\partial}{\partial x}\left[\underline{L}^T\right]\right) + \left(\frac{\partial}{\partial y}\left[\underline{L}\right]\right)\left(\frac{\partial}{\partial y}\left[\underline{L}^T\right]\right) \right\} dA \qquad (75)$$

$$\underline{M} = \int_A \left[\underline{L}\ \underline{L}^T\right] dA \qquad (76)$$

$$\underline{M} = \int_A \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} dA \qquad (77)$$

$$\underline{M} = \int_A \begin{bmatrix} L_1^{\,2} & L_1L_2 & L_1L_3 \\ & L_2^{\,2} & L_2L_3 \\ & & L_3^{\,2} \end{bmatrix} dA \qquad (78)$$

Note that the matrix in equation (78) is represented in upper triangular form since it is symmetric.

$$\int_A L_1^1 L_2^1 \; dA = \frac{1!\,1!\,0!}{(4)!} 2A \tag{79}$$

$$\int_A L_1^1 L_2^1 \; dA = \frac{A}{12} \tag{80}$$

$$\int_A L_1^2 \; dA = \frac{2!\;0!\,0!}{(4)!} 2A \tag{81}$$

$$\int_A L_1^2 \; dA = \frac{2A}{12} \tag{82}$$

$$\underline{M} = \frac{A}{12}\begin{bmatrix} 2 & 1 & 1 \\ & 2 & 1 \\ & & 2 \end{bmatrix} \tag{83}$$

$$N_i = \left\{ \left(\frac{1}{2A}\right)\left[\left(X_j Y_k - X_k Y_j\right) + x\left(Y_j - Y_k\right) + y\left(X_k - X_j\right)\right]\right\} \tag{84}$$

$$\frac{\partial}{\partial x} N_i = \left\{ \left(\frac{1}{2A}\right)\left[Y_j - Y_k\right]\right\} \tag{85}$$

$$\frac{\partial}{\partial y} N_i = \left\{ \left(\frac{1}{2A}\right)\left[X_k - X_j\right]\right\} \tag{86}$$

$$N_j = \left\{ \left(\frac{1}{2A}\right)\left[\left(X_k Y_i - X_i Y_k\right) + x\left(Y_k - Y_i\right) + y\left(X_i - X_k\right)\right]\right\} \tag{87}$$

$$\frac{\partial}{\partial x} N_j = \left\{ \left(\frac{1}{2A} \right) \left[Y_k - Y_i \right] \right\} \tag{88}$$

$$\frac{\partial}{\partial y} N_j = \left\{ \left(\frac{1}{2A} \right) \left[X_i - X_k \right] \right\} \tag{89}$$

$$N_k = \left\{ \left(\frac{1}{2A} \right) \left[\left(X_i Y_j - X_j Y_i \right) + x \left(Y_i - Y_j \right) + y \left(X_j - X_i \right) \right] \right\} \tag{90}$$

$$\frac{\partial}{\partial x} N_k = \left\{ \left(\frac{1}{2A} \right) \left(Y_i - Y_j \right) \right\} \tag{91}$$

$$\frac{\partial}{\partial y} N_k = \left\{ \left(\frac{1}{2A} \right) \left(X_j - X_i \right) \right\} \tag{92}$$

$$\underline{K} = \int_A \left\{ \left(\frac{\partial}{\partial x} \underline{N} \right) \left(\frac{\partial}{\partial x} \underline{N^T} \right) + \left(\frac{\partial}{\partial y} \underline{N} \right) \left(\frac{\partial}{\partial y} \underline{N^T} \right) \right\} dA \tag{93}$$

$$\underline{K} = \int_A \left\{ \left(\frac{\partial}{\partial x} \underline{N} \right) \left(\frac{\partial}{\partial x} \underline{N^T} \right) \right\} dA + \int_A \left\{ \left(\frac{\partial}{\partial y} \underline{N} \right) \left(\frac{\partial}{\partial y} \underline{N^T} \right) \right\} dA \tag{94}$$

$$\underline{K} = \left(\frac{\partial}{\partial x} \underline{N} \right) \left(\frac{\partial}{\partial x} \underline{N^T} \right) \int_A dA + \left(\frac{\partial}{\partial y} \underline{N} \right) \left(\frac{\partial}{\partial y} \underline{N^T} \right) \int_A dA \tag{95}$$

$$\underline{K} = A \left(\frac{\partial}{\partial x} \underline{N} \right) \left(\frac{\partial}{\partial x} \underline{N^T} \right) + A \left(\frac{\partial}{\partial y} \underline{N} \right) \left(\frac{\partial}{\partial y} \underline{N^T} \right) \tag{96}$$

$\underline{K} =$

$$A \begin{bmatrix} \frac{\partial}{\partial x} N_i \\ \frac{\partial}{\partial x} N_j \\ \frac{\partial}{\partial x} N_k \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} N_i & \frac{\partial}{\partial x} N_j & \frac{\partial}{\partial x} N_k \end{bmatrix} + A \begin{bmatrix} \frac{\partial}{\partial y} N_i \\ \frac{\partial}{\partial y} N_j \\ \frac{\partial}{\partial y} N_k \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial y} N_i & \frac{\partial}{\partial y} N_j & \frac{\partial}{\partial y} N_k \end{bmatrix} \tag{97}$$

$$\underline{K} = A \begin{bmatrix} \frac{\partial}{\partial x} N_i \frac{\partial}{\partial x} N_i & \frac{\partial}{\partial x} N_i \frac{\partial}{\partial x} N_j & \frac{\partial}{\partial x} N_i \frac{\partial}{\partial x} N_k \\ & \frac{\partial}{\partial x} N_j \frac{\partial}{\partial x} N_j & \frac{\partial}{\partial x} N_j \frac{\partial}{\partial x} N_k \\ & & \frac{\partial}{\partial x} N_k \frac{\partial}{\partial x} N_k \end{bmatrix}$$

$$+ A \begin{bmatrix} \frac{\partial}{\partial y} N_i \frac{\partial}{\partial y} N_i & \frac{\partial}{\partial y} N_i \frac{\partial}{\partial y} N_j & \frac{\partial}{\partial y} N_i \frac{\partial}{\partial y} N_k \\ & \frac{\partial}{\partial y} N_j \frac{\partial}{\partial y} N_j & \frac{\partial}{\partial y} N_j \frac{\partial}{\partial y} N_k \\ & & \frac{\partial}{\partial y} N_k \frac{\partial}{\partial y} N_k \end{bmatrix} \tag{98}$$

The global matrices can thus be assembled from equations (98) and (83).

References

1. L. Segerlind, Applied Finite Element Analysis, Wiley, New York, 1984.

2. K. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, New Jersey, 1982.

APPENDIX A

Example 1

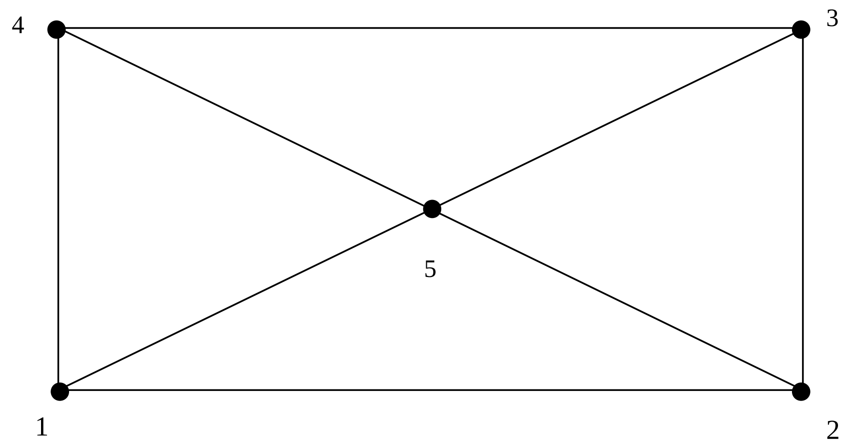

Figure A-1. FE Model

The two-dimensional enclosure has closed boundaries. The dimensions are 20 inch x 10 inch. The model consists of four triangular elements with five nodes.

The global mass matrix is

$$\underline{M} = \begin{bmatrix} 16.667 & 4.167 & 0 & 4.167 & 8.333 \\ & 16.667 & 4.167 & 0 & 8.333 \\ & & 16.667 & 4.167 & 8.333 \\ & & & 16.667 & 8.333 \\ & & & & 33.333 \end{bmatrix}$$

The global stiffness matrix is

$$\underline{K} = \begin{bmatrix} 1.25 & 0.375 & 0 & -0.375 & -1.25 \\ & 1.25 & -0.375 & 0 & -1.25 \\ & & 1.25 & 0.375 & -1.25 \\ & & & 1.25 & -1.25 \\ & & & & 5.00 \end{bmatrix}$$

Each matrix is shown in upper triangular form due to symmetry.

The mass matrix carries a unit of inch^2. The stiffness matrix is non-dimensional.

The generalized eigenvalue problem is

$$\det \{\underline{K} - \lambda\, \underline{M}\} = 0 \qquad \text{(xx)}$$

where

$$\lambda = \left(\frac{\omega}{c} \right)^2 \qquad \text{(xx)}$$

c = speed of sound

c = 13504 in/sec

$\omega = 2\pi f$

The eigenvalues and eigenvectors are found using the Jacobi method in Reference 2.

The natural frequency results are given in Table A-1. The pressure mode shapes are given in Table A-2.

Table A-1. Enclosure 20 inch x 10 inch, FE Results, Four Element Model					
Mode	λ	ω / c	FE Model fn(Hz)	Theoretical fn(Hz)	Error
1	0.000	0.000	0	0	0
2	0.030	0.173	372	336	10.7 %
3	0.120	0.346	745	672	10.9 %
4	0.150	0.387	832	672	23.8 %
5	0.450	0.671	1442	751	92.0 %

Note that the error calculation is made with respect to the theoretical value.

Table A-2. Normalized Pressure Mode Shapes, FE Model					
Node	Mode 1	Mode 2	Mode 3	Mode 4	Mode 4
1	0.447	-0.500	-0.500	-0.500	-0.354
2	0.447	0.500	-0.500	0.500	-0.354
3	0.447	0.500	0.500	-0.500	-0.354
4	0.447	-0.500	0.500	0.500	-0.354
5	0.447	0.000	0.000	0.000	0.707

The FE model mode shapes for the second and third modes are plotted in Figures A-2 and A-4, respectively. The theoretical mode shapes would have a cosine waveform.

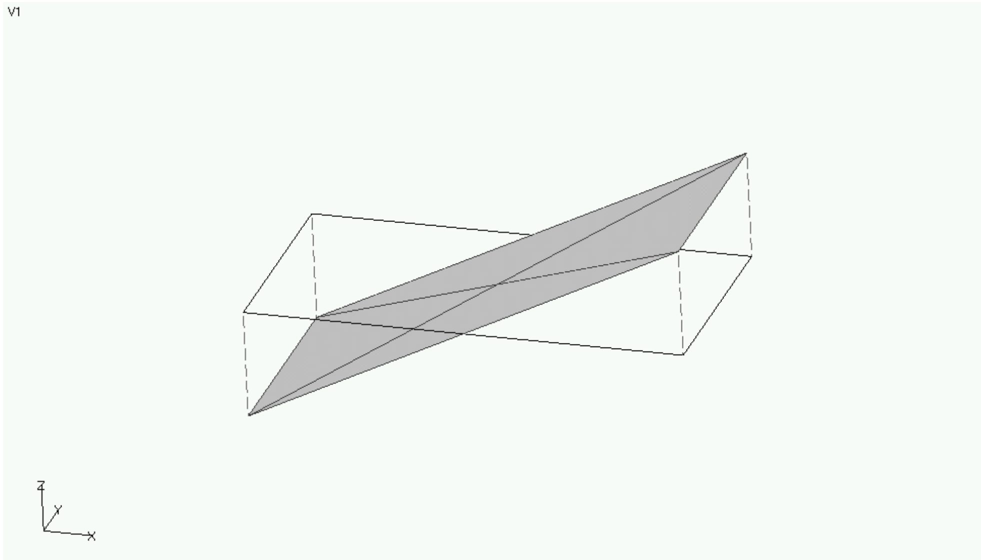

Figure A-2.

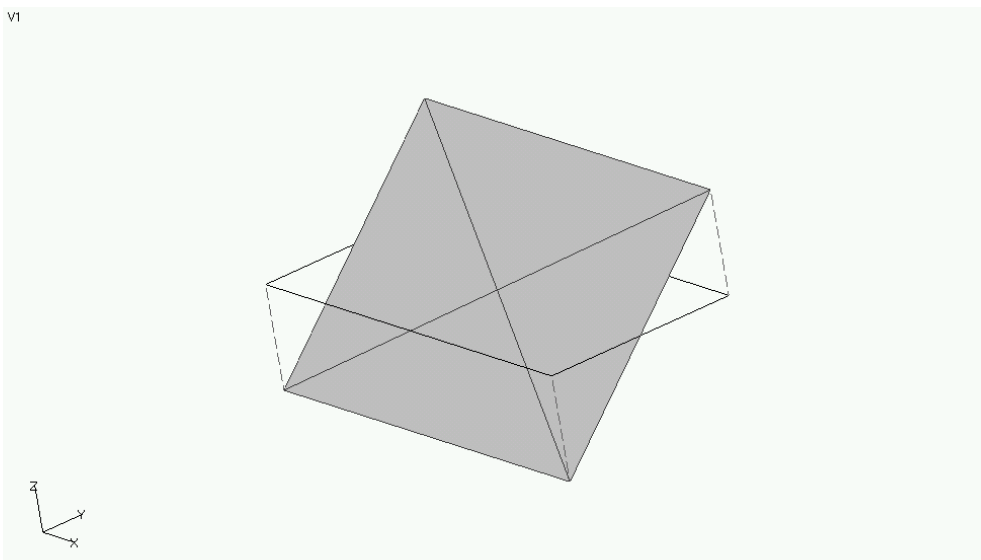

Figure A-3.

APPENDIX B

Example 2

The 20 inch by 10 inch space is analyzed again, with 64 elements and 45 nodes. The model is shown in Figure B-1. The first, ten natural frequencies are shown in Table B-1. The agreement with the respective theoretical values is very good. The fifth mode is shown in Figure B-2.

Table B-1. Enclosure 20 inch x 10 inch, FE Results, 64 Element Model

Mode	FE Model λ	FE Model $\omega./c$	FE Model fn (Hz)	Theoretical fn (Hz)	Error
1	0.000	0.000	0	0	0
2	0.025	0.158	340	336	1.2 %
3	0.104	0.322	692	672	3.0 %
4	0.104	0.322	692	672	3.0 %
5	0.134	0.366	785	751	4.5 %
6	0.226	0.475	1021	950	7.5 %
7	0.249	0.499	1071	1008	6.3 %
8	0.391	0.625	1344	1211	11.0 %
9	0.476	0.690	1483	1344	10.3 %
10	0.478	0.691	1485	1344	10.5 %

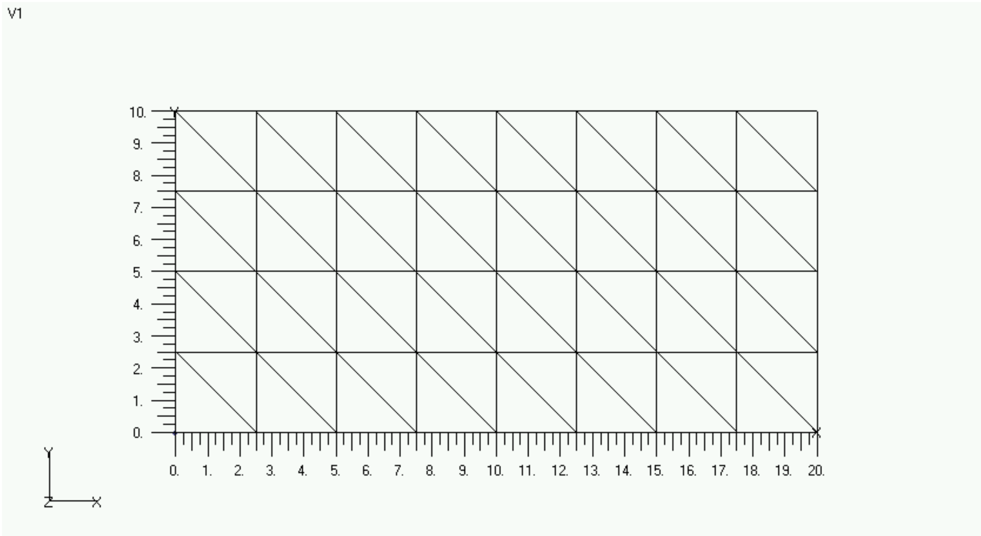

Figure B-1.

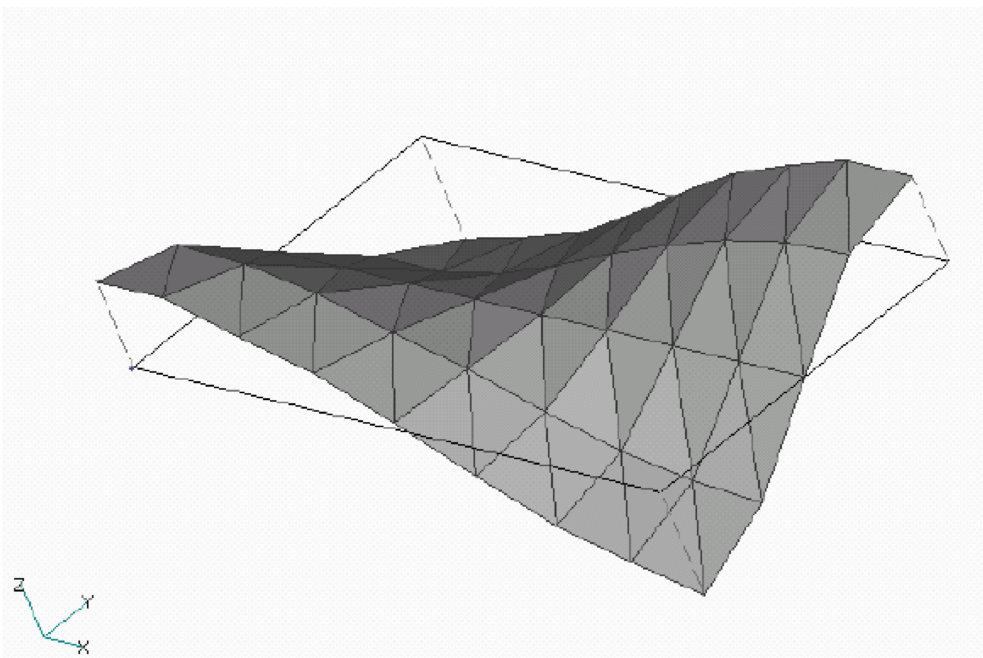

Figure B-2.