THE FUNDAMENTAL FREQUENCY OF AN ISOLATED ANNULAR PLATE
Revision A

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Introduction

The Rayleigh method is used in this tutorial to determine the fundamental bending frequency of an isolated annular plate or bulkhead. The method is taken from References 1 through 3.

A displacement function is assumed which satisfies the geometric boundary conditions. The assumed displacement function is substituted into the strain and kinetic energy equations.

The Rayleigh method gives a natural frequency that is an upper limit of the true natural frequency. The method would give the exact natural frequency if the true displacement function were used. The true displacement function is called an eigenfunction.

Furthermore, Dunkerley's equation is used to determine the lower frequency limit.

Consider the annular plate in Figure 1.

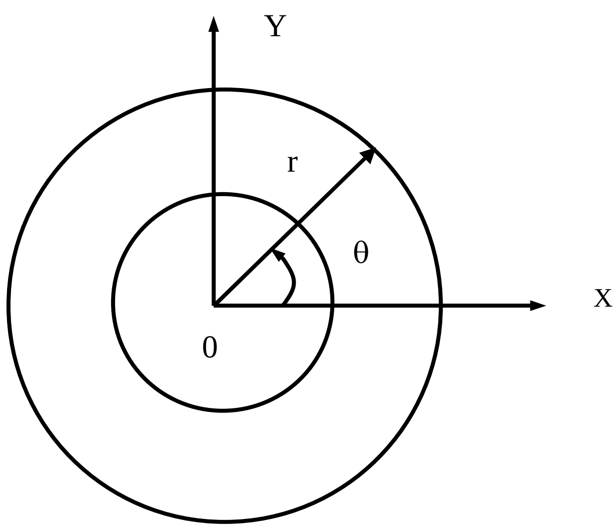

Figure 1.

The plate has an inner radius a and an outer radius b. The displacement perpendicular to the plate is Z. A polar coordinate system is used with the origin at the plate's center of mass.

Consider that the outer rim is supported by isolator mounts. Assume that there are a sufficient number of mounts that the displacement of the outer rim does not vary with theta.

The total strain energy V of the system is

$$V = \frac{D}{2}\int_0^{2\pi}\int_a^b\left[\left(\frac{\partial^2 Z}{\partial r^2}+\frac{1}{r}\frac{\partial Z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 Z}{\partial \theta^2}\right)^2 - 2(1-\mu)\frac{\partial^2 Z}{\partial^2 r}\left(\frac{1}{r}\frac{\partial Z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 Z}{\partial \theta^2}\right)\right.$$

$$\left. + 2(1-\mu)\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial Z}{\partial \theta}\right)\right\}^2\right] r\, dr\, d\theta$$

$$+\frac{1}{2}k\left[Z(b,\theta)\right]^2 \tag{1}$$

The total isolator stiffness is k.

Note that the plate stiffness factor D is given by

$$D = \frac{Eh^3}{12\left(1-\mu^2\right)} \tag{2}$$

where

E = elastic modulus
h = plate thickness
μ = Poisson's ratio

For a displacement which is symmetric about the center,

$$\frac{\partial}{\partial\theta} Z(r,\theta) = 0 \tag{3}$$

$$\frac{\partial^2}{\partial\theta^2} Z(r,\theta) = 0 \tag{4}$$

Substitute equations (3) and (4) into (1).

$$V = \frac{D}{2}\int_0^{2\pi}\int_a^b\left[\left(\frac{\partial^2 Z}{\partial r^2} + \frac{1}{r}\frac{\partial Z}{\partial r}\right)^2 - 2(1-\mu)\frac{\partial^2 Z}{\partial^2 r}\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)\right] r\ dr\, d\theta \ + \ \frac{1}{2}k\left[Z(b,\theta)\right]^2 \tag{5}$$

$$V = \frac{D}{2}\int_0^{2\pi}\int_a^b\left[\left(\frac{\partial^2 Z}{\partial r^2}\right)^2 + 2\left(\frac{\partial^2 Z}{\partial r^2}\right)\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right) + \left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)^2 + (-2+2\mu)\frac{\partial^2 Z}{\partial^2 r}\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)\right] r\ dr\, d\theta$$

$$+\frac{1}{2}k\left[Z(b,\theta)\right]^2 \tag{6}$$

The total strain energy equation for the symmetric case is thus

$$V = \frac{D}{2}\int_0^{2\pi}\int_a^b\left[\left(\frac{\partial^2 Z}{\partial r^2}\right)^2 + \left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)^2 + 2\mu\ \frac{\partial^2 Z}{\partial^2 r}\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)\right] r\ dr\, d\theta \quad + \quad \frac{1}{2}k\left[Z(b,\theta)\right]^2 \tag{7}$$

The total kinetic energy T of the plate bending is given by

$$T = \frac{\rho h \Omega^2}{2}\int_0^{2\pi}\int_a^b Z^2\ r\, dr\, d\theta \tag{8}$$

where

ρ = mass per volume
Ω = angular natural frequency

Isolated Plate

Seek a displacement function that satisfies the geometric boundary conditions.

The geometric boundary conditions are

$$Z(b,\theta) = Z_b \tag{9}$$

$$\left.\frac{\partial^2 Z}{\partial^2 r}\right|_{r=b} = 0 \tag{10}$$

The following function satisfies the geometric boundary conditions.

$$Z(r,\theta) = Z_b + Z_o \cos\left(\frac{\pi r}{2b}\right) \tag{11}$$

where

Z_b is the absolute displacement at the outer rim

Z_o is the displacement of the center of the bulkhead relative to the rim

The partial derivatives are

$$\frac{\partial}{\partial \theta} Z(r,\theta) = 0 \tag{12}$$

$$\frac{\partial^2}{\partial \theta^2} Z(r,\theta) = 0 \tag{13}$$

$$\frac{\partial}{\partial r} Z(r,\theta) = -Z_o\left(\frac{\pi}{2b}\right)\sin\left(\frac{\pi r}{2b}\right) \tag{14}$$

$$\frac{\partial^2}{\partial r^2} Z(r,\theta) = -Z_o \left(\frac{\pi}{2b}\right)^2 \cos\left(\frac{\pi r}{2b}\right) \tag{15}$$

The total kinetic energy T of the plate bending is given by

$$T = \frac{\rho h \Omega^2}{2} \int_0^{2\pi} \int_a^b \left[Z_b + Z_o \cos\left(\frac{\pi r}{2b}\right)\right]^2 r\, dr\, d\theta \tag{16}$$

$$T = \frac{\rho h \Omega^2}{2} \int_0^{2\pi} \int_a^b \left[Z_b + Z_o \cos\left(\frac{\pi r}{2b}\right)\right]\left[Z_b + Z_o \cos\left(\frac{\pi r}{2b}\right)\right] r\, dr\, d\theta \tag{17}$$

$$T = \frac{\rho h \Omega^2}{2} \int_0^{2\pi} \int_a^b \left[Z_b{}^2 + 2 Z_b Z_o \cos\left(\frac{\pi r}{2b}\right) + Z_o{}^2 \left[\cos\left(\frac{\pi r}{2b}\right)\right]^2 \right] r\, dr\, d\theta \tag{18}$$

$$T = \frac{\rho h \Omega^2}{2} \int_0^{2\pi} \int_a^b \left[Z_b{}^2 + 2 Z_b Z_o \cos\left(\frac{\pi r}{2b}\right) + Z_o{}^2 \left[\frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi r}{b}\right)\right] \right] r\, dr\, d\theta \tag{19}$$

$$T = \frac{\rho h \Omega^2}{2} \int_0^{2\pi} \left\{ \frac{1}{2} Z_b{}^2 r^2 + \frac{1}{4} Z_o{}^2 r^2 \right\}\Bigg|_a^b d\theta$$

$$+ \rho h \Omega^2 Z_b Z_o \int_0^{2\pi} \left\{ r \left(\frac{2b}{\pi}\right) \sin\left(\frac{\pi r}{2b}\right) + \left(\frac{2b}{\pi}\right)^2 \cos\left(\frac{\pi r}{2b}\right) \right\}\Bigg|_a^b d\theta$$

$$+ \frac{\rho h \Omega^2}{4} Z_o{}^2 \int_0^{2\pi} \left\{ r \left(\frac{b}{\pi}\right) \sin\left(\frac{\pi r}{b}\right) + \left(\frac{b}{\pi}\right)^2 \cos\left(\frac{\pi r}{b}\right) \right\}\Bigg|_a^b d\theta \tag{20}$$

$$T = \frac{\rho h \Omega^2}{2} \int_0^{2\pi} \left\{ \frac{1}{2} Z_b^2 \left[b^2 - a^2 \right] + \frac{1}{4} Z_o^2 \left[b^2 - a^2 \right] \right\} d\theta$$

$$+ \quad \rho h \Omega^2 Z_b Z_o \int_0^{2\pi} \left\{ \left(\frac{2b^2}{\pi} \right) - \left(\frac{2ab}{\pi} \right) \sin\left(\frac{\pi a}{2b} \right) - \left(\frac{2b}{\pi} \right)^2 \cos\left(\frac{\pi a}{2b} \right) \right\} d\theta$$

$$+ \frac{\rho h \Omega^2}{4} Z_o^2 \int_0^{2\pi} \left\{ -\left(\frac{b}{\pi} \right)^2 - \left(\frac{ab}{\pi} \right) \sin\left(\frac{\pi a}{b} \right) - \left(\frac{b}{\pi} \right)^2 \cos\left(\frac{\pi a}{b} \right) \right\} d\theta$$

(21)

$$T = \frac{\rho h \Omega^2}{2} \left\{ \frac{1}{2} Z_b^2 \left[b^2 - a^2 \right] + \frac{1}{4} Z_o^2 \left[b^2 - a^2 \right] \right\} \int_0^{2\pi} d\theta$$

$$+ \quad \rho h \Omega^2 Z_b Z_o \left\{ \left(\frac{2b^2}{\pi} \right) - \left(\frac{2ab}{\pi} \right) \sin\left(\frac{\pi a}{2b} \right) - \left(\frac{2b}{\pi} \right)^2 \cos\left(\frac{\pi a}{2b} \right) \right\} \int_0^{2\pi} d\theta$$

$$+ \frac{\rho h \Omega^2}{4} Z_o^2 \left\{ -\left(\frac{b}{\pi} \right)^2 - \left(\frac{ab}{\pi} \right) \sin\left(\frac{\pi a}{b} \right) - \left(\frac{b}{\pi} \right)^2 \cos\left(\frac{\pi a}{b} \right) \right\} \int_0^{2\pi} d\theta$$

(22)

$$T = \frac{\rho h \Omega^2 \left[b^2 - a^2 \right]}{8} \left\{ 2\ Z_b^2 + Z_o^2 \right\} \int_0^{2\pi} d\theta$$

$$+ \quad \rho h \Omega^2 Z_b Z_o \left\{ \frac{2b}{\pi} \right\} \left\{ b - a \sin\left(\frac{\pi a}{2b} \right) - \left(\frac{2b}{\pi} \right) \cos\left(\frac{\pi a}{2b} \right) \right\} \int_0^{2\pi} d\theta$$

$$+ \frac{\rho h \Omega^2}{4} Z_o^2 \left\{ \frac{b}{\pi^2} \right\} \left\{ - b - a\pi \sin\left(\frac{\pi a}{b} \right) - b \cos\left(\frac{\pi a}{b} \right) \right\} \int_0^{2\pi} d\theta$$

(23)

$$T = \frac{\rho h \Omega^2 \left[b^2 - a^2 \right]}{8} \left\{ 2\ Z_b^2 + Z_o^2 \right\} 2\pi$$

$$+ \quad \rho h \Omega^2 Z_b Z_o \left\{ \frac{2b}{\pi} \right\} \left\{ b - a \sin\left(\frac{\pi a}{2b} \right) - \left(\frac{2b}{\pi} \right) \cos\left(\frac{\pi a}{2b} \right) \right\} 2\pi$$

$$+ \frac{\rho h \Omega^2}{4} Z_o^2 \left\{ \frac{b}{\pi^2} \right\} \left\{ - b - a\pi \sin\left(\frac{\pi a}{b} \right) - b \cos\left(\frac{\pi a}{b} \right) \right\} 2\pi$$

(24)

$$T = \frac{\rho h \Omega^2 \pi \left[b^2 - a^2 \right]}{4} \left\{ 2\ Z_b^2 + Z_o^2 \right\}$$

$$+ \quad 4b \rho h \Omega^2 Z_b Z_o \left\{ b - a \sin\left(\frac{\pi a}{2b} \right) - \left(\frac{2b}{\pi} \right) \cos\left(\frac{\pi a}{2b} \right) \right\}$$

$$+ \frac{\rho h \Omega^2}{2} Z_o^2 \left\{ \frac{b}{\pi} \right\} \left\{ - b - a\pi \sin\left(\frac{\pi a}{b} \right) - b \cos\left(\frac{\pi a}{b} \right) \right\}$$

(25)

$$T = \frac{\rho h \Omega^2 \pi \left[b^2 - a^2\right]}{4}\left\{2\ Z_b{}^2 + Z_o{}^2\right\}$$

$$+ \quad 4b\rho h \Omega^2 Z_b Z_o \left\{ b - a\sin\left(\frac{\pi a}{2b}\right) - \left(\frac{2b}{\pi}\right)\cos\left(\frac{\pi a}{2b}\right)\right\}$$

$$+ Z_o{}^2 \left\{\frac{b^2 \rho h \Omega^2}{2}\left[-1 - \cos\left(\frac{\pi a}{b}\right)\right] + \frac{\rho h \Omega^2 b^2 \pi}{4}\right\}$$

(26)

$$T = \frac{\rho h \Omega^2 \pi \left[b^2 - a^2\right]}{4}\left\{2\ Z_b{}^2 + Z_o{}^2\right\}$$

$$+ \quad 4b\rho h \Omega^2 Z_b Z_o \left\{ b - a\sin\left(\frac{\pi a}{2b}\right) - \left(\frac{2b}{\pi}\right)\cos\left(\frac{\pi a}{2b}\right)\right\}$$

$$+ \ \frac{\rho h \Omega^2}{2} Z_o{}^2 \left\{\frac{b}{\pi}\right\}\left\{- b - a\pi \sin\left(\frac{\pi a}{b}\right) - b\cos\left(\frac{\pi a}{b}\right)\right\}$$

(27)

Again, the total strain energy for the symmetric case is

$$V = \frac{D}{2}\int_0^{2\pi}\int_a^b \left[\left(\frac{\partial^2 Z}{\partial r^2}\right)^2 + \left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)^2 + \ 2\mu \frac{\partial^2 Z}{\partial^2 r}\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)\right] r\ dr\, d\theta \ + \ \frac{1}{2} k \left[Z(b,\theta)\right]^2$$

(28)

$$V = \frac{D}{2}\int_0^{2\pi}\int_a^b \left[\left(\frac{\partial^2 Z}{\partial r^2}\right)^2\right] r\, dr\, d\theta \quad + \quad \frac{D}{2}\int_0^{2\pi}\int_a^b \left[\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)^2\right] r\, dr\, d\theta$$
$$+ \frac{D}{2}\int_0^{2\pi}\int_a^b \left[\ 2\mu\, \frac{\partial^2 Z}{\partial^2 r}\left(\frac{1}{r}\frac{\partial Z}{\partial r}\right)\right] r\, dr\, d\theta \quad + \quad \frac{1}{2} k\left[Z(b,\theta)\right]^2 \tag{29}$$

$$V = \quad + \frac{D}{2}\int_0^{2\pi}\int_a^b Z_o{}^2 \left[\left(\frac{\pi}{2b}\right)^4 \cos^2\left(\frac{\pi r}{2b}\right)\right] r\, dr\, d\theta$$
$$+ \frac{D}{2}\int_0^{2\pi}\int_a^b Z_o{}^2 \left[\frac{1}{r^2}\left(\frac{\pi}{2b}\right)^2 \sin^2\left(\frac{\pi r}{2b}\right)\right] r\, dr\, d\theta$$
$$+ \frac{D}{2}\int_0^{2\pi}\int_a^b Z_o{}^2 \left[(2\mu)\left(\frac{\pi}{2b}\right)^3 \frac{1}{r}\cos\left(\frac{\pi r}{2b}\right)\sin\left(\frac{\pi r}{2b}\right)\right] r\, dr\, d\theta \ + \ \frac{1}{2} k\, Z_b{}^2 \tag{30}$$

$$V = \quad + \frac{Z_o{}^2 D}{2}\int_0^{2\pi}\int_a^b \left[\left(\frac{\pi}{2b}\right)^4 \cos^2\left(\frac{\pi r}{2b}\right)\right] r\, dr\, d\theta$$
$$+ \frac{Z_o{}^2 D}{2}\int_0^{2\pi}\int_a^b \left[\frac{1}{r^2}\left(\frac{\pi}{2b}\right)^2 \sin^2\left(\frac{\pi r}{2b}\right)\right] r\, dr\, d\theta$$
$$+ \frac{Z_o{}^2 D}{2}\int_0^{2\pi}\int_a^b \left[(2\mu)\left(\frac{\pi}{2b}\right)^3 \frac{1}{r}\cos\left(\frac{\pi r}{2b}\right)\sin\left(\frac{\pi r}{2b}\right)\right] r\, dr\, d\theta \ + \ \frac{1}{2} k\, Z_b{}^2 \tag{31}$$

$$V = \; + \frac{Z_o{}^2 D}{2}\left(\frac{\pi}{2b}\right)^4 \int_0^{2\pi}\int_a^b \left[\cos^2\left(\frac{\pi r}{2b}\right)\right] r \, dr \, d\theta$$

$$+ \frac{Z_o{}^2 D}{2}\left(\frac{\pi}{2b}\right)^2 \int_0^{2\pi}\int_a^b \left[\frac{1}{r^2}\sin^2\left(\frac{\pi r}{2b}\right)\right] r \, dr \, d\theta$$

$$+ \frac{Z_o{}^2 D}{2}(2\mu)\left(\frac{\pi}{2b}\right)^3 \int_0^{2\pi}\int_a^b \left[\frac{1}{r}\cos\left(\frac{\pi r}{2b}\right)\sin\left(\frac{\pi r}{2b}\right)\right] r \, dr \, d\theta \; + \; \frac{1}{2} k Z_b{}^2 \tag{32}$$

$$V = \; + \frac{Z_o{}^2 \, D}{2}\left(\frac{\pi}{2b}\right)^4 \left(\frac{1}{2}\right) \int_0^{2\pi}\int_a^b \left[1 + \cos\left(\frac{\pi r}{b}\right)\right] r \, dr \, d\theta$$

$$+ \frac{Z_o{}^2 \, D}{2}\left(\frac{\pi}{2b}\right)^2 \int_0^{2\pi}\int_a^b \left[\frac{1}{r}\sin^2\left(\frac{\pi r}{2b}\right)\right] dr \, d\theta$$

$$+ \frac{Z_o{}^2 \, D}{2}\left(\mu\left(\frac{\pi}{2b}\right)^3\right) \int_0^{2\pi}\int_a^b \left[\sin\left(\frac{\pi r}{b}\right)\right] dr \, d\theta \; + \; \frac{1}{2} k Z_b{}^2 \tag{33}$$

The first and third integrals are evaluating using the tables in Appendix A.

$$V = +\frac{Z_o^2\ D}{2}\left(\frac{\pi}{2b}\right)^4\left(\frac{1}{2}\right)\int_0^{2\pi}\left[\frac{r^2}{2}+\frac{b\,r}{\pi}\sin\left(\frac{\pi\,r}{b}\right)+\frac{b^2}{\pi^2}\cos\left(\frac{\pi\,r}{b}\right)\right]\Bigg|_b^b\ d\theta$$

$$+\frac{Z_o^2\ D}{2}\left(\frac{\pi}{2b}\right)^2\int_0^{2\pi}\frac{1}{2}\left[\frac{x^2}{2\cdot 2!}-\frac{x^4}{4\cdot 4!}+\frac{x^6}{6\cdot 6!}-\frac{x^8}{8\cdot 8!}+\frac{x^{10}}{10\cdot 10!}-\frac{x^{12}}{12\cdot 12!}\right]\Bigg|_{a\pi/b}^{\pi}\ d\theta$$

$$-\frac{Z_o^2\ D}{2}\left(\mu\left(\frac{\pi}{2b}\right)^3\right)\left(\frac{b}{\pi}\right)\int_0^{2\pi}\cos\left(\frac{\pi\,r}{b}\right)\Bigg|_a^b d\theta\ +\ \frac{1}{2}k\,Z_b^2 \tag{34}$$

$$V = +\frac{Z_o^2\ D}{2}\left(\frac{\pi}{2b}\right)^4\left(\frac{1}{2}\right)\int_0^{2\pi}\left\{\left[\frac{b^2}{2}-\frac{b^2}{\pi^2}\right]-\left[\frac{a^2}{2}+\frac{a\,b}{\pi}\sin\left(\frac{\pi\,a}{b}\right)+\frac{b^2}{\pi^2}\cos\left(\frac{\pi\,a}{b}\right)\right]\right\}d\theta$$

$$+\frac{Z_o^2\ D}{2}\left(\frac{\pi}{2b}\right)^2\int_0^{2\pi}\left\{0.688\left(\frac{a}{b}\right)^3-1.57\left(\frac{a}{b}\right)^2+0.0552\left(\frac{a}{b}\right)+0.823\right\}d\theta$$

$$+\frac{Z_o^2\ D}{2}\left(\mu\left(\frac{\pi}{2b}\right)^3\right)\left(\frac{2b}{\pi}\right)\int_0^{2\pi}\left\{-1-\cos\left(\frac{\pi\,a}{b}\right)\right\}\ d\theta\ +\ \frac{1}{2}k\,Z_b^2 \tag{35}$$

$$V = \; + \frac{Z_o{}^2\ D}{2}\left(\frac{\pi}{2b}\right)^4\left(\frac{1}{2}\right)\ \left\{\left[\frac{b^2}{2} - \frac{b^2}{\pi^2}\right] - \left[\frac{a^2}{2} + \frac{a\,b}{\pi}\sin\left(\frac{\pi\,a}{b}\right) + \frac{b^2}{\pi^2}\cos\left(\frac{\pi\,a}{b}\right)\right]\right\}\int_0^{2\pi} d\theta$$

$$+ \frac{Z_o{}^2\ D}{2}\left(\frac{\pi}{2b}\right)^2\left\{0.688\left(\frac{a}{b}\right)^3 -1.57\left(\frac{a}{b}\right)^2 + 0.0552\left(\frac{a}{b}\right) + 0.823\right\}\int_0^{2\pi} d\theta$$

$$+ \frac{Z_o{}^2\ D}{2}\left(\mu\left(\frac{\pi}{2b}\right)^2\right)\left\{-1 - \cos\left(\frac{\pi\,a}{b}\right)\right\}\int_0^{2\pi} d\theta \; + \; \frac{1}{2}k\,Z_b{}^2 \tag{36}$$

$$V = \; + \frac{Z_o{}^2\ D}{2}\left(\frac{\pi}{2b}\right)^4\left(\frac{1}{2}\right)\ \left\{\left[\frac{b^2}{2} - \frac{b^2}{\pi^2}\right] - \left[\frac{a^2}{2} + \frac{a\,b}{\pi}\sin\left(\frac{\pi\,a}{b}\right) + \frac{b^2}{\pi^2}\cos\left(\frac{\pi\,a}{b}\right)\right]\right\}2\pi$$

$$+ \frac{Z_o{}^2\ D}{2}\left(\frac{\pi}{2b}\right)^2\left\{0.688\left(\frac{a}{b}\right)^3 -1.57\left(\frac{a}{b}\right)^2 + 0.0552\left(\frac{a}{b}\right) + 0.823\right\}2\pi$$

$$+ \frac{Z_o{}^2\ D}{2}\left(\mu\left(\frac{\pi}{2b}\right)^2\right)\left\{-1 - \cos\left(\frac{\pi\,a}{b}\right)\right\}2\pi \; + \; \frac{1}{2}k\,Z_b{}^2 \tag{37}$$

$$V = \; + Z_o{}^2\ D\pi\left(\frac{\pi}{2b}\right)^4\left(\frac{1}{2}\right)\ \left\{\left[\frac{b^2}{2} - \frac{b^2}{\pi^2}\right] - \left[\frac{a^2}{2} + \frac{a\,b}{\pi}\sin\left(\frac{\pi\,a}{b}\right) + \frac{b^2}{\pi^2}\cos\left(\frac{\pi\,a}{b}\right)\right]\right\}$$

$$+ Z_o{}^2\ D\pi\left(\frac{\pi}{2b}\right)^2\left\{0.688\left(\frac{a}{b}\right)^3 -1.57\left(\frac{a}{b}\right)^2 + 0.0552\left(\frac{a}{b}\right) + 0.823\right\}$$

$$+ Z_o{}^2\ D\pi\left(\mu\left(\frac{\pi}{2b}\right)^2\right)\left\{-1 - \cos\left(\frac{\pi\,a}{b}\right)\right\} \; + \; \frac{1}{2}k\,Z_b{}^2 \tag{38}$$

Now equate the total kinetic energy with the total strain energy per Rayleigh's method.

$$\frac{\rho h\,\Omega^2 \pi\left[b^2 - a^2\right]}{4}\left\{2\,Z_b{}^2 + Z_o{}^2\right\} \quad + \quad 4b\rho h\,\Omega^2 Z_b Z_o\left\{b - a\sin\left(\frac{\pi a}{2b}\right) - \left(\frac{2b}{\pi}\right)\cos\left(\frac{\pi a}{2b}\right)\right\}$$

$$+\ \frac{\rho h\,\Omega^2}{2} Z_o{}^2\left\{\frac{b}{\pi}\right\}\left\{-b - a\pi\sin\left(\frac{\pi a}{b}\right) - b\cos\left(\frac{\pi a}{b}\right)\right\}$$

$$= \ + Z_o{}^2\ D\pi\left(\frac{\pi}{2b}\right)^4\left(\frac{1}{2}\right)\ \left\{\left[\frac{b^2}{2} - \frac{b^2}{\pi^2}\right] - \left[\frac{a^2}{2} + \frac{a\,b}{\pi}\sin\left(\frac{\pi a}{b}\right) + \frac{b^2}{\pi^2}\cos\left(\frac{\pi a}{b}\right)\right]\right\}$$

$$+ Z_o{}^2\ D\pi\left(\frac{\pi}{2b}\right)^2\left\{0.688\left(\frac{a}{b}\right)^3 - 1.57\left(\frac{a}{b}\right)^2 + 0.0552\left(\frac{a}{b}\right) + 0.823\right\}$$

$$+ Z_o{}^2\ D\pi\left(\mu\left(\frac{\pi}{2b}\right)^2\right)\ \left\{-1 - \cos\left(\frac{\pi a}{b}\right)\right\}\ +\ \frac{1}{2}k\,Z_b{}^2$$

(51)

Let

$$Z_o = \beta\, Z_b \qquad (52)$$

$$\frac{\rho h \Omega^2 \pi\left[b^2 - a^2\right]}{4}\left\{2\ Z_b{}^2 + \beta^2\ Z_b{}^2\right\} \quad + \quad 4b\rho h\Omega^2\beta\ Z_b{}^2\left\{b - a\sin\left(\frac{\pi a}{2b}\right) - \left(\frac{2b}{\pi}\right)\cos\left(\frac{\pi a}{2b}\right)\right\}$$

$$+\ \frac{\rho h\Omega^2}{2}\beta^2\ Z_b{}^2\left\{\frac{b}{\pi}\right\}\left\{-b - a\pi\sin\left(\frac{\pi a}{b}\right) - b\cos\left(\frac{\pi a}{b}\right)\right\}$$

$$= \quad +\beta^2\ Z_b{}^2\ D\pi\left(\frac{\pi}{2b}\right)^4\left(\frac{1}{2}\right)\quad\left\{\left[\frac{b^2}{2} - \frac{b^2}{\pi^2}\right] - \left[\frac{a^2}{2} + \frac{a\,b}{\pi}\sin\left(\frac{\pi a}{b}\right) + \frac{b^2}{\pi^2}\cos\left(\frac{\pi a}{b}\right)\right]\right\}$$

$$+\beta^2\ Z_b{}^2\ D\pi\left(\frac{\pi}{2b}\right)^2\left\{0.688\left(\frac{a}{b}\right)^3 - 1.57\left(\frac{a}{b}\right)^2 + 0.0552\left(\frac{a}{b}\right) + 0.823\right\}$$

$$+\beta^2\ Z_b{}^2\ D\pi\left(\mu\left(\frac{\pi}{2b}\right)^2\right)\left\{-1 - \cos\left(\frac{\pi a}{b}\right)\right\} \ +\ \frac{1}{2}k\,Z_b{}^2$$

(51)

$$\frac{\rho h \Omega^2 \pi \left[b^2 - a^2 \right]}{4} \left\{ 2 + \beta^2 \right\} \quad + \quad 4b\rho h \Omega^2 \beta \left\{ b - a \sin\left(\frac{\pi a}{2b} \right) - \left(\frac{2b}{\pi} \right) \cos\left(\frac{\pi a}{2b} \right) \right\}$$

$$+ \frac{\rho h \Omega^2}{2} \beta^2 \left\{ \frac{b}{\pi} \right\} \left\{ -b - a\pi \sin\left(\frac{\pi a}{b} \right) - b \cos\left(\frac{\pi a}{b} \right) \right\}$$

$$= \; + \beta^2 \; D\pi \left(\frac{\pi}{2b} \right)^4 \left(\frac{1}{2} \right) \left\{ \left[\frac{b^2}{2} - \frac{b^2}{\pi^2} \right] - \left[\frac{a^2}{2} + \frac{a\,b}{\pi} \sin\left(\frac{\pi a}{b} \right) + \frac{b^2}{\pi^2} \cos\left(\frac{\pi a}{b} \right) \right] \right\}$$

$$+ \beta^2 \; D\pi \left(\frac{\pi}{2b} \right)^2 \left\{ 0.688 \left(\frac{a}{b} \right)^3 - 1.57 \left(\frac{a}{b} \right)^2 + 0.0552 \left(\frac{a}{b} \right) + 0.823 \right\}$$

$$+ \beta^2 \; D\pi \left(\mu \left(\frac{\pi}{2b} \right)^2 \right) \left\{ -1 - \cos\left(\frac{\pi a}{b} \right) \right\} \; + \; \frac{1}{2} k \tag{52}$$

$$\rho h \Omega^2 \left\{ \left(\frac{\pi}{4}\right) \left[b^2 - a^2 \right] \left[2 + \beta^2 \right] \quad + \quad 4b\beta \left\{ b - a \sin\left(\frac{\pi a}{2b}\right) - \left(\frac{2b}{\pi}\right) \cos\left(\frac{\pi a}{2b}\right) \right\} \right.$$

$$\left. + \ \beta^2 \left\{ \frac{b}{2\pi} \right\} \left\{ -b - a\pi \sin\left(\frac{\pi a}{b}\right) - b \cos\left(\frac{\pi a}{b}\right) \right\} \ \right\}$$

$$= \ + \beta^2 \ D\pi \left(\frac{\pi}{2b}\right)^4 \left(\frac{1}{2}\right) \ \left\{ \left[\frac{b^2}{2} - \frac{b^2}{\pi^2} \right] - \left[\frac{a^2}{2} + \frac{a\,b}{\pi} \sin\left(\frac{\pi a}{b}\right) + \frac{b^2}{\pi^2} \cos\left(\frac{\pi a}{b}\right) \right] \right\}$$

$$+ \beta^2 \ D\pi \left(\frac{\pi}{2b}\right)^2 \left\{ 0.688 \left(\frac{a}{b}\right)^3 - 1.57 \left(\frac{a}{b}\right)^2 + 0.0552 \left(\frac{a}{b}\right) + 0.823 \right\}$$

$$+ \beta^2 \ D\pi \left(\mu \left(\frac{\pi}{2b}\right)^2 \right) \left\{ -1 - \cos\left(\frac{\pi a}{b}\right) \right\} \ + \ \frac{1}{2} k \tag{53}$$

$$\rho h \Omega^2 \left\{ \left(\frac{\pi}{4}\right) \left[b^2 - a^2 \right] \left[2 + \beta^2 \right] \;+\; 4b\beta \left\{ b - a\sin\left(\frac{\pi a}{2b}\right) - \left(\frac{2b}{\pi}\right)\cos\left(\frac{\pi a}{2b}\right) \right\} \right.$$

$$\left. + \beta^2 \left\{\frac{b}{2\pi}\right\} \left\{ - b - a\pi\sin\left(\frac{\pi a}{b}\right) - b\cos\left(\frac{\pi a}{b}\right) \right\} \right\}$$

$$= \beta^2 \left(\frac{D\pi^3}{4b^2}\right) \left\{ \left(\frac{\pi}{2b}\right)^2 \left(\frac{1}{2}\right) \left\{ b^2 \left[\frac{1}{2} - \frac{1}{\pi^2} \right] - \left[\frac{a^2}{2} + \frac{a\,b}{\pi}\sin\left(\frac{\pi a}{b}\right) + \frac{b^2}{\pi^2}\cos\left(\frac{\pi a}{b}\right) \right] \right\} \right.$$

$$+ \left\{ 0.688\left(\frac{a}{b}\right)^3 - 1.57\left(\frac{a}{b}\right)^2 + 0.0552\left(\frac{a}{b}\right) + 0.823 \right\}$$

$$\left. + \mu \left\{ -1 - \cos\left(\frac{\pi a}{b}\right) \right\} \right\} + \frac{1}{2}k \tag{54}$$

The goal is to select the value of β that yields the lowest value of Ω. This may be done by trial-and-error. Again, this approach gives an estimate of the upper limit of the natural frequency.

The lower limit fn is estimated via Dunkerley's equation.

$$\frac{1}{fn^2} \approx \frac{1}{f_1^{\,2}} + \frac{1}{f_2^{\,2}} \tag{55}$$

where

f_1 is the spring-mass frequency with the bulkhead as a rigid-mass

f_2 is the frequency of the bulkhead simply-supported at its outer rim without isolation springs

Strictly speaking, this is a rather tenuous use of Dunkerley's equation since the mass of the bulkhead is counted twice. On the other hand, it properly accounts for the combined stiffness of the isolator springs and the bulkhead itself acting in series.

Example

An annular honeycomb bulkhead has an outer diameter of 37.5 inches. The inner diameter is 18.75 inch.

The total mass including the added avionics components is 136 lbm, or 4.23 slugs, or 0.352 lbf sec^2/in.

A diagram of the honeycomb plate cross-section is shown in Figure 2. The face sheets and core are aluminum.

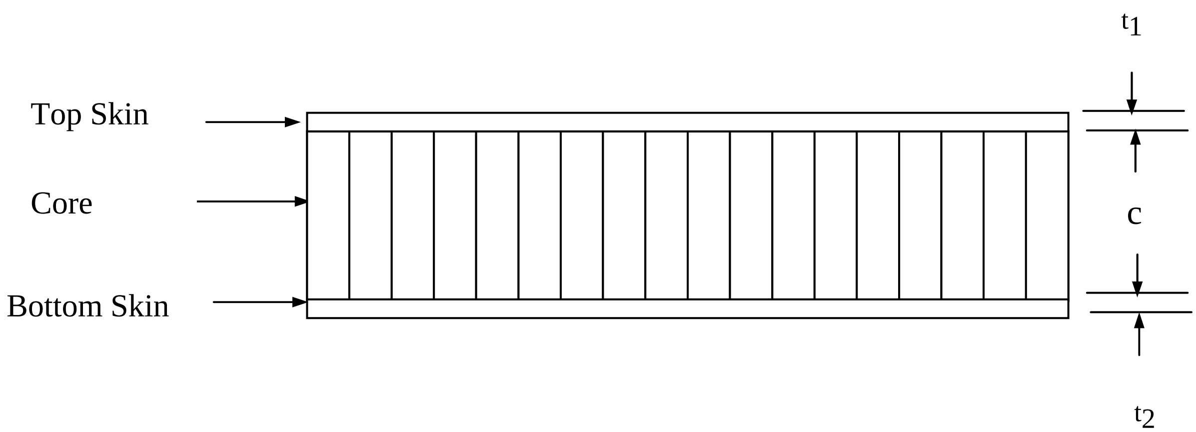

Figure 2.

Let

D = bending stiffness

E_f = elastic modulus of skin facings

t_1 = thickness of top skin

t_2 = thickness of bottom skin

c = honeycomb core thickness

μ = Poisson’s ratio

Assume

1. The skin elasticity is much greater than the core elasticity.
2. Each skin has the same material.
3. Each skin is "thin" relative to the core.

The bending stiffness is

$$D = \left[\frac{E_f}{1-\mu^2}\right]\left[\frac{t_1\, t_2\, H^2}{t_1 + t_2}\right] \tag{55}$$

where

$$H = c + \frac{1}{2}\left[t_1 + t_2\right] \tag{56}$$

Now assume that each face sheet thickness is equal to t.

$$D = \left[\frac{E_f}{1-\mu^2}\right]\left[\frac{t\, H^2}{2}\right] \tag{57}$$

$$H = c + t \tag{58}$$

$$D = \left[\frac{E_f}{1-\mu^2}\right]\left[\frac{t\,(c+t)^2}{2}\right] \tag{59}$$

For the sample problem,

T = 1 in

c = 1.126 in

t = 0.062 in

The natural frequency is found using the Matlab program given in Appendix B.

The output is:

```
>> annular_elastic

 annular_elastic.m   ver 1.1   June 20, 2005

 by Tom Irvine
 Email:  tomirvine@aol.com

 This program calculates the fundamental frequency of a
 annular honeycomb bulkhead supported by isolators
 mounted along the outside rim.

 Assume that the face sheets have the same thickness.


 Enter outside diameter (inch)
 37.5

 Enter inside diameter (inch)
 18.75

 Enter core thickness (inch)
 1.125

 Enter individual face sheet thickness (inch)
 0.062

 Enter total mass (lbm)
 136

 Select face sheet material
 1=aluminum  2=other
 1

 Enter the total isolator stiffness (lbf/in)
 25524

 Bending Stiffness/Elastic Modulus =     0.048 in^3
```

Bending Stiffness = 4.8e+005 lbf in^2

Total Thickness = 1.249 in

Volume = 1035 in^3

spring mass fn = 42.84 Hz

annular plate simply-supported outside, free inside

fn = 76.88 Hz

Dunkerley fn = 37.42 Hz

Rayleigh fn = 40.12 Hz

Rayleigh Eigenvector = [1 1.42]

Final Frequency Estimate: 37.42 Hz < fn < 40.12 Hz

(End of output)

The first term in the eigenvector is the absolute displacement at the rim, which is 1. The second term is the absolute displacement of the center of the bulkhead, which is 1.42.

The upper frequency estimate is obtained via Rayleigh's method. The lower estimate is calculated via Dunkerley's equation.

The bulkhead behaves as both a simply supported plate and a spring-mass system. The estimated natural frequency due to these combined effects is approximately 38.8 Hz, taking the average of the upper and lower bounds.

Note that the estimated frequency agrees within 2% of the finite element results for this bulkhead, as shown in Figure 3.

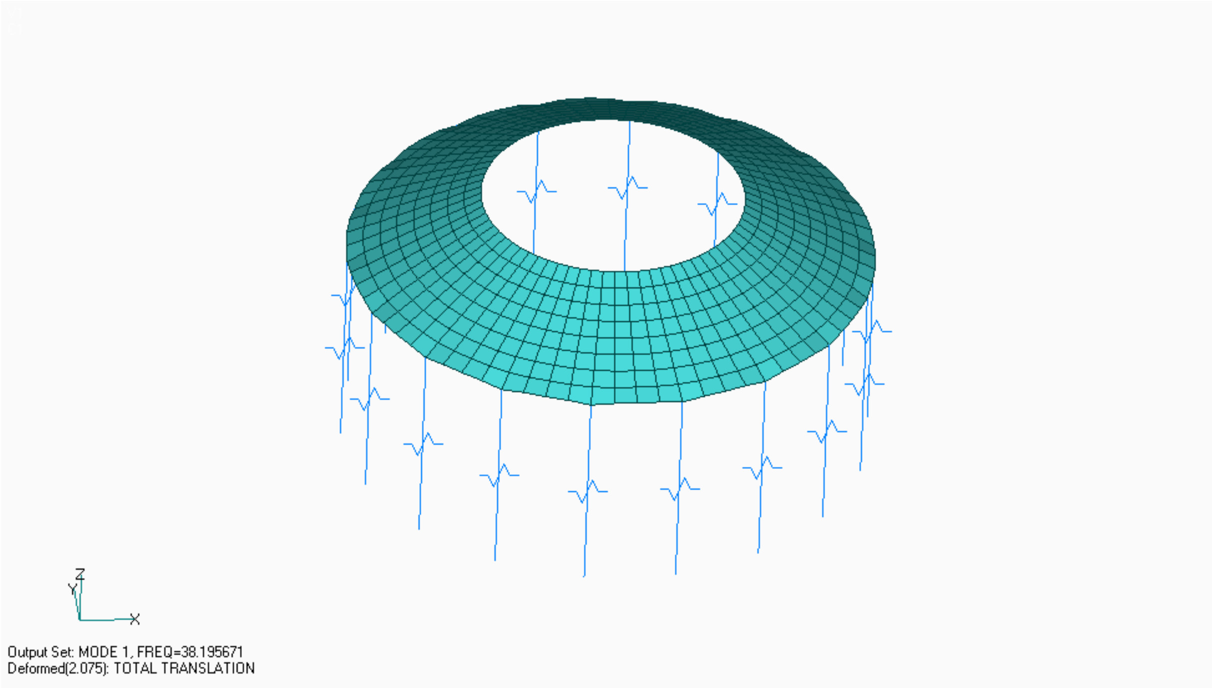

Figure 3. Mode 1, File: annular_bulkhead.mod, fn = 38.2 Hz

References

1. Dave Steinberg, Vibration Analysis for Electronic Equipment, Wiley-Interscience, New York, 1988.

2. Weaver, Timoshenko, and Young; Vibration Problems in Engineering, Wiley-Interscience, New York, 1990.

3. Arthur W. Leissa, Vibration of Plates, NASA SP-160, National Aeronautics and Space Administration, Washington D.C., 1969.

4. Jan Tuma, Engineering Mathematics Handbook, McGraw-Hill, New York, 1979.

APPENDIX A

Integrals

Equation (A-1) is taken from Reference 1.

$$\int x \cos bx \, dx = \frac{x \sin bx}{b} + \frac{\cos bx}{b^2} \tag{A-1}$$

Now consider

$$\int_a^b \frac{1}{r} \sin^2\left(\frac{\pi r}{2b}\right) dr = \frac{1}{2}\int_a^b \frac{1}{r}\left[1 - \cos\left(\frac{\pi r}{b}\right)\right] dr \tag{A-2}$$

Nondimensionalize,

$$x = \frac{\pi r}{b} \tag{A-3}$$

$$\frac{b}{\pi} x = r \tag{A-4}$$

$$dx = \frac{\pi}{b} dr \tag{A-5}$$

$$\frac{b}{\pi} dx = dr \tag{A-6}$$

$$\int_a^b \frac{1}{r} \sin^2\left(\frac{\pi r}{2a}\right) dr = \frac{b}{2\pi}\left(\frac{\pi}{b}\right)\int_{a\pi/b}^{\pi} \frac{1}{x}[1 - \cos x] \, dx \tag{A-7}$$

$$\int_a^b \frac{1}{r} \sin^2\left(\frac{\pi r}{2b}\right) dr = \frac{1}{2}\int_{a\pi/b}^{\pi} \frac{1}{x}[1 - \cos x] \, dx \tag{A-8}$$

Recall the series

$$\cos x \approx 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\frac{x^{12}}{12!} \tag{A-9}$$

$$\int_a^b \frac{1}{r}\sin^2\left(\frac{\pi r}{2b}\right)dr \approx \frac{1}{2}\int_{a\pi/b}^{\pi}\frac{1}{x}\left\{1-\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\frac{x^{12}}{12!}\right]\right\}dx \tag{A-10}$$

$$\int_a^b \frac{1}{r}\sin^2\left(\frac{\pi r}{2b}\right)dr \approx \frac{1}{2}\int_{a\pi/b}^{\pi}\frac{1}{x}\left[\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\frac{x^8}{8!}+\frac{x^{10}}{10!}-\frac{x^{12}}{12!}\right]dx \tag{A-11}$$

$$\int_a^b \frac{1}{r}\sin^2\left(\frac{\pi r}{2b}\right)dr \approx \frac{1}{2}\int_{a\pi/b}^{\pi}\left[\frac{x^1}{2!}-\frac{x^3}{4!}+\frac{x^5}{6!}-\frac{x^7}{8!}+\frac{x^9}{10!}-\frac{x^{11}}{12!}\right]dx \tag{A-12}$$

$$\int_a^b \frac{1}{r}\sin^2\left(\frac{\pi r}{2b}\right)dr \approx \frac{1}{2}\left[\frac{x^2}{2\cdot 2!}-\frac{x^4}{4\cdot 4!}+\frac{x^6}{6\cdot 6!}-\frac{x^8}{8\cdot 8!}+\frac{x^{10}}{10\cdot 10!}-\frac{x^{12}}{12\cdot 12!}\right]\Bigg|_{a\pi/b}^{\pi} \tag{A-13}$$

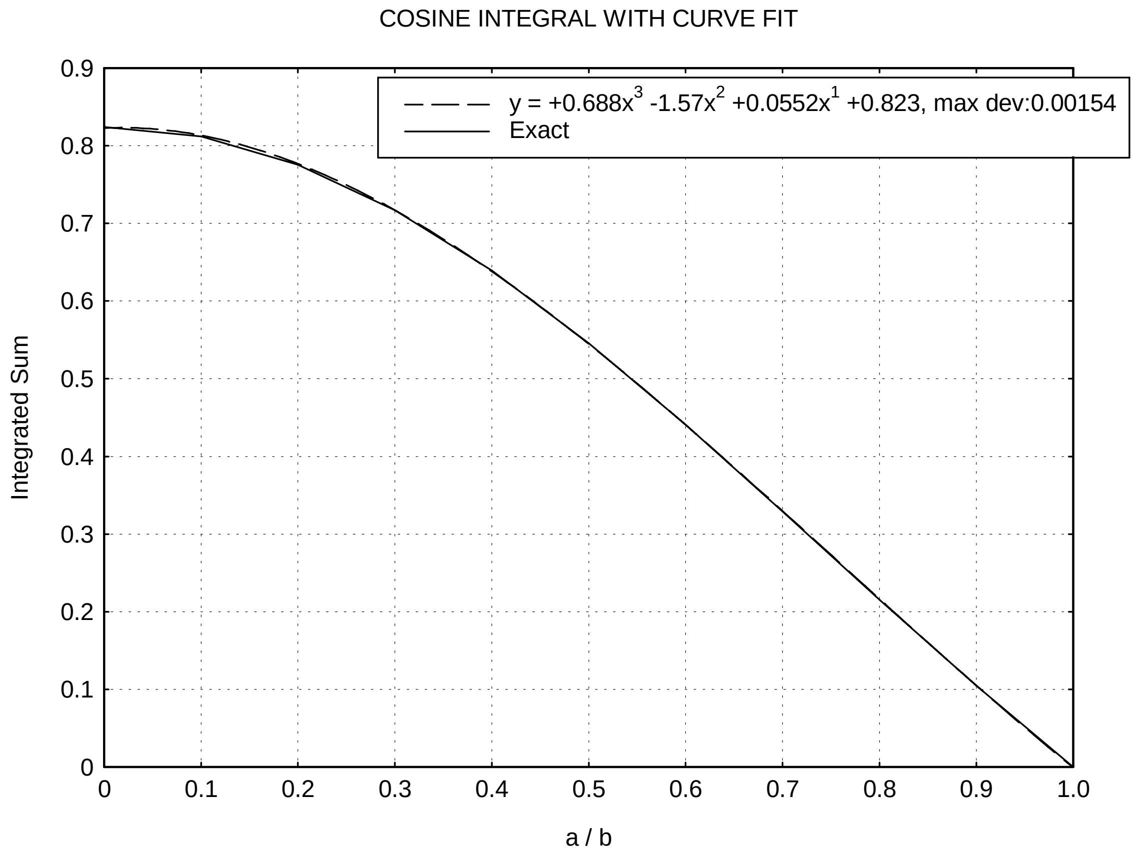

Figure A-1.

Equation (A-13) is plotted in Figure A-1. A curve-fit is used to determine the approximate result in equation (A-14).

$$\int_a^b \frac{1}{r}\sin^2\left(\frac{\pi r}{2b}\right) dr \approx 0.688\left(\frac{a}{b}\right)^3 - 1.57\left(\frac{a}{b}\right)^2 + 0.0552\left(\frac{a}{b}\right) + 0.823 , \quad \text{for } a < b \tag{A-14}$$

APPENDIX B

Matlab Script

```
disp('  ');
disp(' annular_elastic.m   ver 1.3   May 16, 2007  ');
disp('  ');
disp(' by Tom Irvine');
disp(' revised by Matthew Fifield 5/16/07');
disp(' Email:  tomirvine@aol.com ');
disp('  ');
disp(' This program calculates the fundamental frequency of a ');
disp(' annular honeycomb bulkhead supported by isolators');
disp(' mounted along the outside rim.');
disp('  ');
disp(' Assume that the face sheets have the same thickness.');
disp('  ');
%
clear a;
clear b;
clear beta;
clear D;
clear t;
clear rho;
clear mass;
clear volume;
%
iflag=0;
while(iflag==0)
   disp(' ');
   disp(' Enter outside diameter (inch)')
   diam=input(' ');
   b=diam/2.;
%
   disp(' ');
   disp(' Enter inside diameter (inch)')
   diam=input(' ');
   a=diam/2.;
%
   if(a>=b)
      disp(' ')
      disp(' Input data error.  O.D. must be > I.D. ');
   else
      iflag=1;
   end
%
```

```
   if(a<0)
       disp(' Input data error. ');
       iflag=0;
   end
end
%
disp(' ');
disp(' Enter core thickness (inch)')
c=input(' ');
%
disp(' ');
disp(' Enter individual face sheet thickness (inch)')
t=input(' ');
%
disp(' ');
disp(' Enter total mass (lbm)');
mass=input(' ');
mass=mass/386.;
%
disp(' ');
disp(' Select face sheet material ');
disp(' 1=aluminum  2=other ');
imat = input(' ');
%
if(imat==1)
    E=1.0e+07;
else
    disp(' ');
    disp(' Enter elastic modulus of face sheets (lbf/in^2)');
    E=input(' ');
end
%
disp(' ');
disp(' Enter the total isolator stiffness (lbf/in) ');
k=input(' ');
%
mu=0.3;
%
H=c+t;
D=E/(1-mu^2);
D=(D*t*H^2)/2;
%
thickness = c+2*t;
volume = pi*( b^2 - a^2 )*thickness;
rho = mass/volume;
%
```

```
out1= sprintf('\n Bending Stiffness/Elastic Modulus = %9.4g in^3',D/E);
disp(out1);
%
out1= sprintf('\n Bending Stiffness = %9.4g lbf in^2 ',D);
disp(out1);
%
out1= sprintf('\n Total Thickness = %9.4g in ',thickness);
disp(out1);
%
out1= sprintf('\n Volume = %9.4g in^3 \n',volume);
disp(out1);
%
fsm=sqrt(k/(rho*volume))/(2*pi);
%
out1= sprintf('\n spring mass fn = %9.4g Hz ',fsm);
disp(out1);
%
adb=a/b;
padb=pi*adb;
%
scale= 0.00194*exp(9.79*adb) +4.82 -0.0479*adb;
num=D;
den=(rho*thickness);
fssp=scale*(1/b^2)*sqrt(num/den)/(2*pi);
%
out1= sprintf('\n annular plate simply-supported outside, free inside \n\n fn = %9.4g
Hz\n',fssp);
disp(out1);
%
fdu=(1/fssp^2)+(1/fsm^2);
fdu=1./fdu;
fdu=sqrt(fdu);
out1= sprintf(' Dunkerley fn = %9.4g Hz\n',fdu);
disp(out1);
%
minimum=1.0e+90;
min_beta=0.;
%
for( i=0:10000)
   beta=i/500;
%
   d1=(pi/4)*(b^2-a^2)*(2+beta^2);
%
   d2=b-a*sin((padb)/2.)-((2*b)/pi)*cos((padb)/2.);
   d2=d2*4*b*beta;
%
```

```
   d3=-b-a*pi*sin(padb)-b*cos((padb));
   d3=d3*beta^2*b/(2*pi);
%
   den=d1+d2+d3;
   den=den*rho*thickness;
%
   n11=b^2*((1/2)-(1/pi^2));
   n12=(((a^2)/2)+(a*b/pi)*sin(padb)+(b^2/pi^2)*cos(padb));
   n1=n11+n12;
   n1=n1*(0.5*(pi/(2*b))^2);
   n2=0.688*adb^3 -1.57*adb^2 +0.0552*adb +0.823;
   n3=mu*(-1-cos(padb));
%
   num=n1+n2+n3;
   num=num*beta^2*D*pi^3/(4*b^2);
   num=num+0.5*k;
%
   om2=num/den;
%
   if(om2 < minimum )
      minimum=om2;
      min_beta=beta;
   end
%
end
%
fn=sqrt(minimum)/(2*pi);
%
out1= sprintf('\n Rayleigh fn = %9.4g Hz\n',fn);
disp(out1);
%
disp('');
%
out1= sprintf(' Rayleigh Eigenvector = [ 1 %7.3g] \n',(1+min_beta));
disp(out1);
%
if(fdu<fn)
   out1= sprintf(' Final Frequency Estimate: %7.4g Hz < fn < %7.4g Hz \n',fdu,fn);
else
   out1= sprintf(' Final Frequency Estimate: %7.4g Hz < fn < %7.4g Hz \n',fn,fdu);
end
disp(out1);
```