SAMPLE AUTOMOBILE VIBRATION PROBLEM
Revision A

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Introduction

The following example is taken from Reference 1. Consider the automobile in Figure 1. The translation and rotation are both referenced to the center-of-gravity. The vehicle is modeled as a two-degree-of-freedom system as shown in Figure 2.

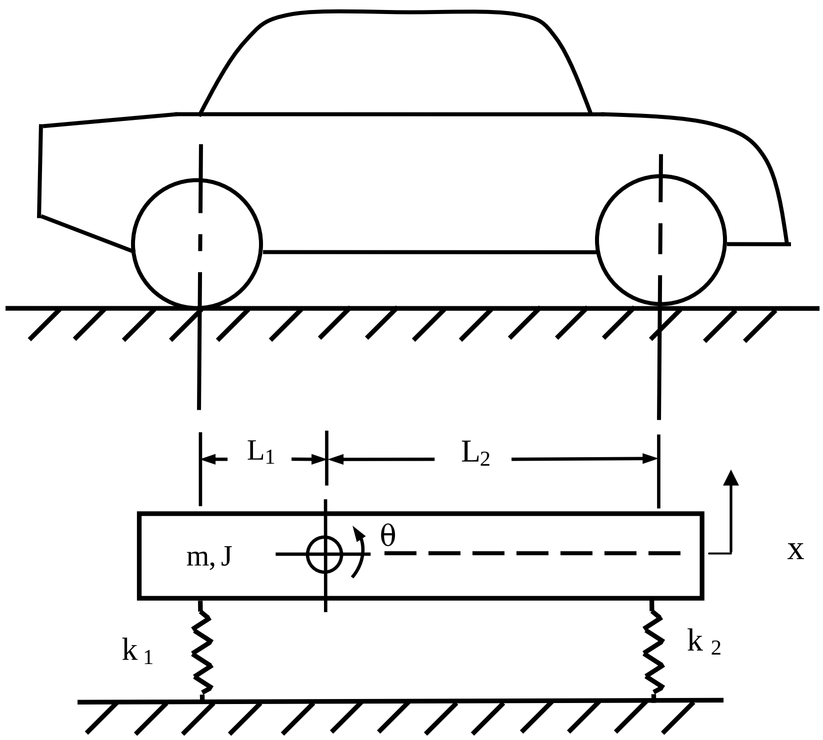

Figure 1.

Variables

m	mass
J	mass moment of inertia about the C.G.
k	spring stiffness
L	length from spring attachment to C.G.

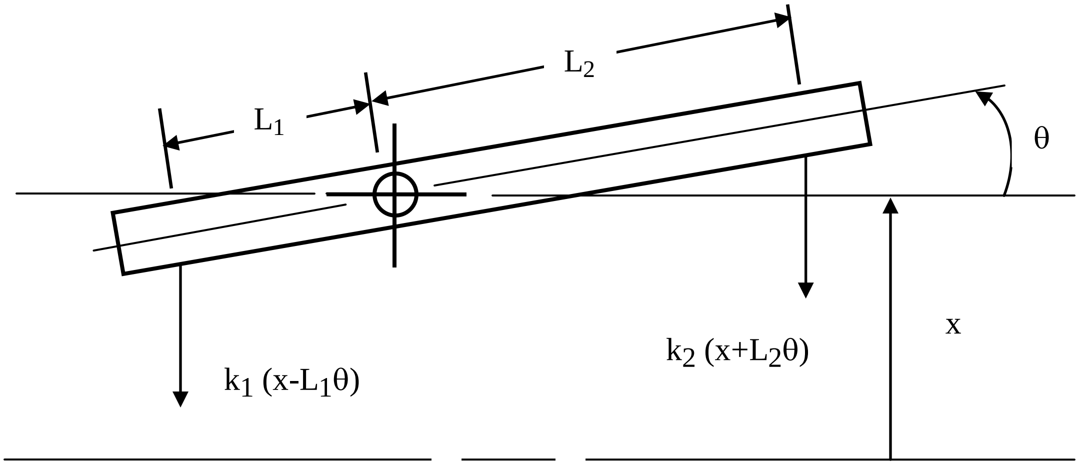

Figure 2.

Sign Convention:

Translation: upward in vertical axis is positive.
Rotation: counter-clockwise is positive.

Assume small angular displacement.

Sum the forces in the vertical direction.

$$\sum F = m\ddot{x} \tag{1}$$

$$m\ddot{x} = -k_1(x - L_1\theta) - k_2(x + L_2\theta) \tag{2}$$

$$m\ddot{x} + k_1(x - L_1\theta) + k_2(x + L_2\theta) = 0 \tag{3}$$

$$m\ddot{x} + k_1x - k_1L_1\theta + k_2x + k_2L_2\theta = 0 \tag{4}$$

$$m\ddot{x} + (k_1 + k_2)x + (-k_1L_1 + k_2L_2)\theta = 0 \tag{5}$$

Sum the moments about the center of mass.

$$\sum M = J\ddot{\theta} \tag{6}$$

$$J\ddot{\theta} = k_1 L_1(x - L_1\theta) - k_2 L_2(x + L_2\theta) \tag{7}$$

$$J\ddot{\theta} - k_1 L_1(x - L_1\theta) + k_2 L_2(x + L_2\theta) = 0 \tag{8}$$

$$J\ddot{\theta} - k_1 L_1 x + k_1 L_1^2\theta + k_2 L_2 x + k_2 L_2^2\theta = 0 \tag{9}$$

$$J\ddot{\theta} + \left(k_1 L_1^2 + k_2 L_2^2\right)\theta + \left(-k_1 L_1 + k_2 L_2\right)x = 0 \tag{10}$$

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_1 L_1 + k_2 L_2 \\ -k_1 L_1 + k_2 L_2 & k_1 L_1^2 + k_2 L_2^2 \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11}$$

The vibration modes have translation and rotation which will be coupled if

$$k_1 L_1 \neq k_2 L_2 \tag{12}$$

Note that the moment of inertia J is

$$J = m r^2 \tag{13}$$

where r is the radius of gyration.

Eigenvalues and Eigenvectors

Equation (11) is coupled via the stiffness matrix. An intermediate goal is to decouple the equation.

Simplify,

$$M\ddot{\bar{x}} + K\ddot{\bar{x}} = 0 \tag{14}$$

where

$$M = \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \tag{15}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_1 L_1 + k_2 L_2 \\ -k_1 L_1 + k_2 L_2 & k_1 L_1^2 + k_2 L_2^2 \end{bmatrix} \tag{16}$$

$$\bar{x} = \begin{bmatrix} x \\ 0 \end{bmatrix} \tag{17}$$

Seek a solution of the form

$$\bar{x} = \bar{q} \exp(j\omega t) \tag{18}$$

The q vector is the generalized coordinate vector.

Note that

$$\dot{\bar{x}} = j\omega \bar{q} \exp(j\omega t) \tag{19}$$

$$\ddot{\bar{x}} = -\omega^2 \bar{q} \exp(j\omega t) \tag{20}$$

Substitute equations (18) through (20) into equation (14).

$$-\omega^2 M \bar{q} \exp(j\omega t) + K \exp(j\omega t) = \bar{0} \tag{21}$$

$$\left\{ -\omega^2 M \bar{q} + K\bar{q} \right\} \exp(j\omega t) = \bar{0} \tag{22}$$

$$-\omega^2 M \bar{q} + K\bar{q} = \bar{0} \tag{23}$$

$$\left\{ -\omega^2 M + K \right\} \bar{q} = \bar{0} \tag{24}$$

$$\left\{ K - \omega^2 M \right\} \bar{q} = \bar{0} \tag{25}$$

Equation (25) is an example of a generalized eigenvalue problem. The eigenvalues can be found by setting the determinant equal to zero.

$$\det \left\{ K - \omega^2 M \right\} = 0 \tag{26}$$

$$\det \left\{ \begin{bmatrix} k_1 + k_2 & -k_1 L_1 + k_2 L_2 \\ -k_1 L_1 + k_2 L_2 & k_1 L_1^2 + k_2 L_2^2 \end{bmatrix} - \omega^2 \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \right\} = 0 \tag{27}$$

$$\det\begin{bmatrix}(k_1+k_2)-\omega^2 m & -k_1 L_1 + k_2 L_2 \\ -k_1 L_1 + k_2 L_2 & \left(k_1 L_1^2 + k_2 L_2^2\right)-\omega^2 J\end{bmatrix} = 0 \qquad (28)$$

Table 1. Parameters	
Variable	Value
M	3500 lbm = 108.8 slugs = 108.8 lbf sec^2 /ft
L_1	4.4 ft
L_2	5.6 ft
k_1	2000 lbf / ft
k_2	2400 lbf / ft
R	4.0 ft
J	1741 slugs ft^2 = 1741 lbf ft sec^2

$$k_1 + k_2 \; = \; 2000 \text{ lbf / ft} + 2400 \text{ lbf / ft} \qquad (29)$$

$$k_1 + k_2 = 4400 \text{ lbf / ft} \qquad (30)$$

$$-k_1 L_1 + k_2 L_2 = \left[2000\frac{\text{lbf}}{\text{ft}}\right][4.4\,\text{ft}] + \left[2400\frac{\text{lbf}}{\text{ft}}\right][5.6\,\text{ft}] \qquad (31)$$

$$-k_1 L_1 + k_2 L_2 = 4640 \text{ lbf} \qquad (32)$$

$$\left(k_1 L_1^2 + k_2 L_2^2\right) = \left[2000\frac{\text{lbf}}{\text{ft}}\right][4.4\,\text{ft}]^2 + \left[2400\frac{\text{lbf}}{\text{ft}}\right][5.6\,\text{ft}]^2 \qquad (33)$$

$$\left(k_1 L_1^2 + k_2 L_2^2\right) = 1.140\text{e}+05 \text{ lbf ft} \qquad (34)$$

$$\det\begin{bmatrix} 4400\,\frac{\text{lbf}}{\text{ft}} - \omega^2\left(108.8\,\frac{\text{lbf sec}^2}{\text{ft}}\right) & 4640\,\text{lbf} \\ 4640\,\text{lbf} & \left(1.140\text{e}+05\ \ \text{lbf ft}\right) - \omega^2\left(1741\,\text{lbf ft sec}^2\right) \end{bmatrix} = 0 \qquad (35)$$

The units are consistent. Omit the units for brevity.

$$\det\begin{bmatrix} 4400 - 108.8\,\omega^2 & 4640 \\ 4640 & \left(1.140\text{e}+05\right) - 1741\omega^2 \end{bmatrix} = 0 \qquad (36)$$

$$\left\{4400 - 108.8\,\omega^2\right\}\left\{\left(1.140\text{e}+05\right) - 1741\omega^2\right\} - 4640^2 = 0 \qquad (37)$$

$$\left(1.894\text{e}+05\right)\omega^4 - \left(2.006\text{e}+07\right)\omega^2 + \left(5.016\text{e}+08\right) - 4640^2 = 0 \qquad (38)$$

$$\left(1.894\text{e}+05\right)\omega^4 - \left(2.006\text{e}+07\right)\omega^2 + \left(4.801\text{e}+08\right) = 0 \qquad (39)$$

$$1.894\,\omega^4 - 200.6\,\omega^2 + 4801 = 0 \qquad (40)$$

The eigenvalues are the roots of the polynomial.

$$\omega_1^2 = \frac{200.6 - \sqrt{200.6^2 = 4(1.894)(4801)}}{2(1.894)} \qquad (41)$$

$$\omega_2^2 = \frac{200.6 + \sqrt{200.6^2 = 4(1.894)(4801)}}{2(1.894)} \qquad (42)$$

$$\omega_1^2 = \frac{200.6 - 62.19}{3.79} \qquad (43)$$

$$\omega_2^2 = \frac{200.6 + 62.19}{3.79} \qquad (44)$$

$$\omega_1^2 = 36.517\left(\text{rad}/\text{sec}\right)^2 \qquad (45)$$

$$\omega_2^2 = 69.404 (\mathrm{rad/sec})^2 \quad (46)$$

$$\omega_1 = 6.043 \ (\mathrm{rad/sec}) \quad (47)$$

$$\omega_2 = 8.331 \ (\mathrm{rad/sec}) \quad (48)$$

$$f_1 = 0.962 \text{ Hz} \quad (49)$$

$$f_2 = 1.33 \text{ Hz} \quad (50)$$

The eigenvectors are found via the following equations.

$$\left\{ K - \omega_1{}^2 M \right\} \bar{q}_1 = \bar{0} \quad (51)$$

$$\left\{ K - \omega_2{}^2 M \right\} \bar{q}_2 = \bar{0} \quad (52)$$

where

$$\bar{q}_1 = \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} \quad (53)$$

$$\bar{q}_2 = \begin{bmatrix} q_{21} \\ q_{22} \end{bmatrix} \quad (54)$$

Note that the first component in each vector is a translation and that the second component is a rotation.

Now solve for the first eigenvector.

$$\begin{bmatrix} 4400 - 108.8(36.517) & 4640 \\ 4640 & (1.140\mathrm{e}+05) - 1741(36.517) \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} = 0 \quad (55)$$

$$\begin{bmatrix} 426.62 & 4640 \\ 4640 & 50419 \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} = 0 \quad (56)$$

$$\begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} = \begin{bmatrix} 1.000 \\ -0.092 \end{bmatrix} \quad (57)$$

Now solve for the second eigenvector.

$$\begin{bmatrix} 4400-108.8(69.404) & 4640 \\ 4640 & (1.140e+05)-1741(69.404) \end{bmatrix} \begin{bmatrix} q_{21} \\ q_{22} \end{bmatrix} = 0 \tag{58}$$

$$\begin{bmatrix} -3151 & 4640 \\ 4640 & -6832 \end{bmatrix} \begin{bmatrix} q_{21} \\ q_{22} \end{bmatrix} = 0 \tag{59}$$

$$\begin{bmatrix} q_{21} \\ q_{22} \end{bmatrix} = \begin{bmatrix} 1.000 \\ 0.679 \end{bmatrix} \tag{60}$$

The first and second mode shapes are plotted in Figures 3 and 4, respectively.

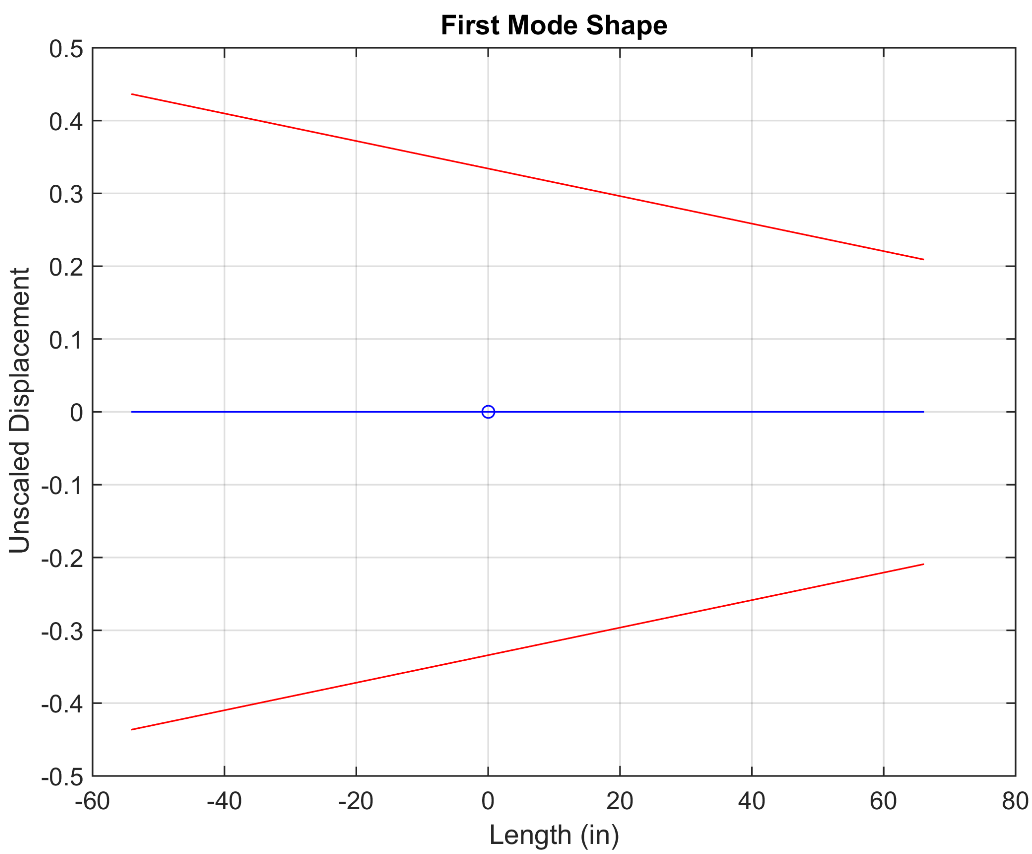

Figure 3.

The blue curve is the undeformed mode shape. The circle indicates the C.G.

The red curves are the mode shapes.

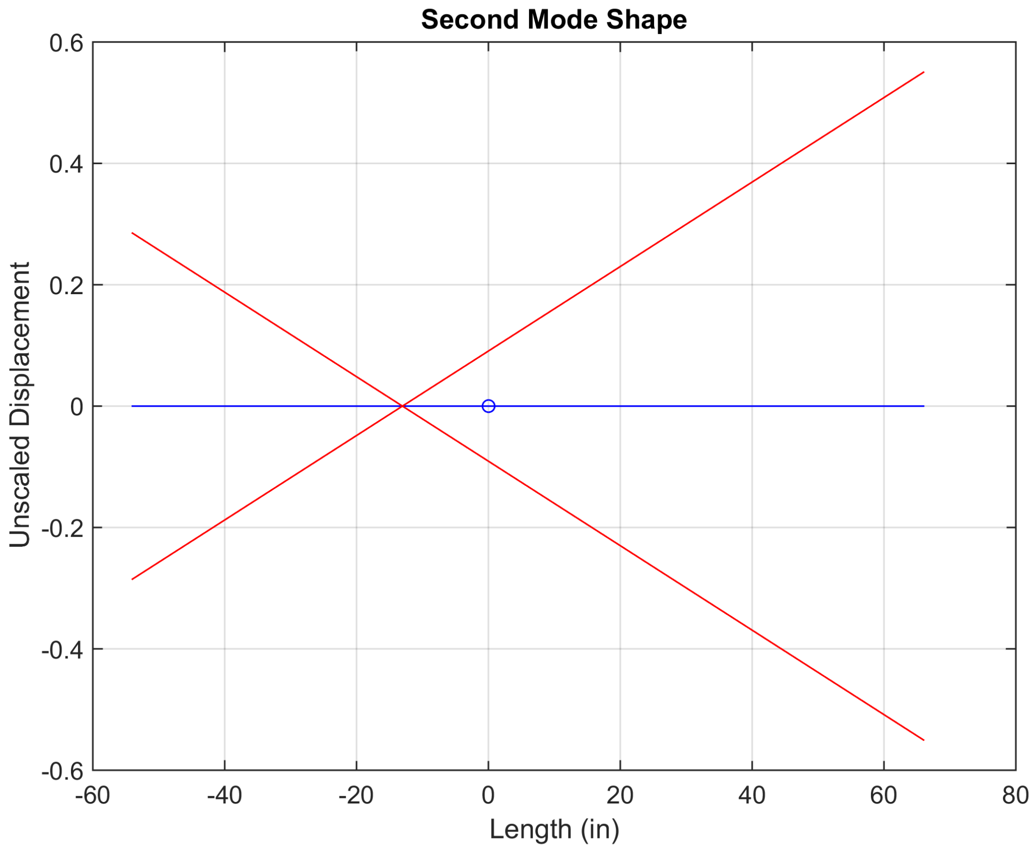

Figure 4.

The blue curve is the undeformed mode shape. The circle indicates the C.G.

The red curves are the mode shapes.

$$\begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} = \begin{bmatrix} 1.000 \\ -0.092 \end{bmatrix} \tag{61}$$

The displacement d_1 as a function of the length u is

$$d_1\,(u) = u \tan(-\,0.092) + 1 \tag{62}$$

$$d_1(u) = -\,0.092\,u + 1 \tag{63}$$

Recall the second eigenvector.

$$\begin{bmatrix} q_{21} \\ q_{22} \end{bmatrix} = \begin{bmatrix} 1.000 \\ 0.679 \end{bmatrix} \tag{64}$$

The nodal point is found by setting the displacement equal to zero. The nodal point for the first mode is thus 10.87 ft forward of the center of mass.

The displacement d_2 as a function of the length u is

$$d_2(u) = u \tan(0.679) + 1 \tag{65}$$

$$d_2(u) = 0.807\,u + 1 \tag{66}$$

The nodal point for the second mode is thus 1.24 ft aft of the center of mass.

<u>Reference</u>

1. W. Thomson, Theory of Vibration with Applications, Second Edition, Prentice-Hall, New Jersey, 1981.