TRANSVERSE VIBRATION OF A CANTILEVER BEAM SUBJECTED TO A CONSTANT AXIAL LOAD VIA THE FINITE ELEMENT METHOD

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Theory

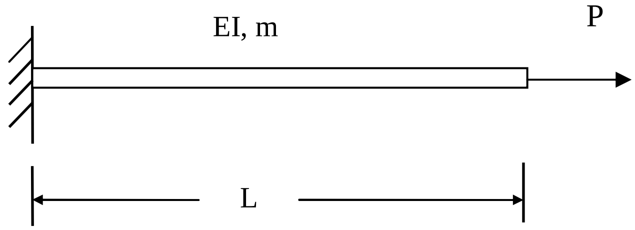

Figure 1.

The governing differential equation for the transverse displacement y(x, t) of cantilever beam subject to an axial load applied at its free end is

$$\frac{\partial^2}{\partial x^2}\left\{EI(x)\frac{d^2}{\partial x^2}y(x,t)\right\}+\frac{\partial}{\partial x}\left[P\frac{\partial}{\partial x}y(x,t)\right]+m\frac{\partial^2 y(x,t)}{\partial t^2}=0 \tag{1}$$

where

E	is the modulus of elasticity
I	is the area moment of inertia
m	is the mass per length
L	is the length
P	is the axial tension load

Equation (1) is taken from Reference 1.

Assume that the load P is constant.

$$\frac{\partial^2}{\partial x^2}\left\{ EI(x) \frac{d^2}{\partial x^2} y(x,t) \right\} + P \frac{d^2}{\partial x^2} y(x,t) + m \frac{\partial^2 y(x,t)}{\partial t^2} = 0 \tag{2}$$

The product EI is the bending stiffness.

Assume that the solution of equation (2) is separable in time and space.

$$y(x,t) = Y(x) f(t) \tag{3}$$

$$\frac{\partial^2}{\partial x^2}\left\{ EI(x) \frac{\partial^2}{\partial x^2} Y(x) f(t) \right\} + P \left[\frac{\partial^2}{\partial x^2} Y(x) f(t) \right] = -m \frac{\partial^2 Y(x) f(t)}{\partial t^2} \tag{4}$$

$$f(t)\left\{ \frac{\partial^2}{\partial x^2}\left\{ EI(x) \frac{\partial^2}{\partial x^2} Y(x) \right\} + P \left[\frac{\partial^2}{\partial x^2} Y(x) \right] \right\} = -Y(x)\, m \frac{\partial^2 f(t)}{\partial t^2} \tag{5}$$

The partial derivatives change to ordinary derivatives.

$$f(t)\left\{ \frac{d^2}{dx^2}\left\{ EI(x) \frac{d^2}{dx^2} Y(x) \right\} + P_o \left[\frac{d^2}{dx^2} Y(x) \right] \right\} = -Y(x)\, m \frac{d^2 f(t)}{dt^2} \tag{6}$$

$$\frac{1}{m\,Y(x)}\left\{\frac{d^2}{dx^2}\left\{EI\frac{d^2}{dx^2}Y(x)\right\}+P\left[\frac{d^2}{dx^2}Y(x)\right]\right\}=-\frac{1}{f(t)}\frac{d^2 f(t)}{dt^2}$$

(7)

The left-hand side of equation (7) depends on x only. The right hand side depends on t only. Both x and t are independent variables. Thus equation (7) only has a solution if both sides are constant. Let ω^2 be the constant.

$$\frac{1}{m\,Y(x)}\left\{\frac{d^2}{dx^2}\left\{EI\frac{d^2}{dx^2}Y(x)\right\}+P\left[\frac{d^2}{dx^2}Y(x)\right]\right\}$$

$$= -\frac{1}{f(t)}\frac{d^2 f(t)}{dt^2} = \omega^2$$

(8)

Equation (8) yields two independent equations.

$$\frac{d^2}{dx^2}\left\{EI\frac{d^2}{dx^2}Y(x)\right\}+P\left[\frac{d^2}{dx^2}Y(x)\right]-m\,\omega^2\,Y(x)=0$$

(9)

$$\frac{d^2}{dt^2}\,f(t)+\omega^2 f(t)=0 \qquad (10)$$

Equation (9) is a homogeneous, forth order, ordinary differential equation.

The weighted residual method is applied to equation (9). This method is suitable for boundary value problems. An alternative method would be the energy method.

There are numerous techniques for applying the weighted residual method. Specifically, the Galerkin approach is used in this tutorial.

The differential equation (9) is multiplied by a test function $\phi(x)$. Note that the test function $\phi(x)$ must satisfy the homogeneous essential boundary conditions. The essential boundary conditions are the prescribed values of Y and its first derivative.

The test function is not required to satisfy the differential equation, however.

The product of the test function and the differential equation is integrated over the domain. The integral equation is set to zero.

$$\int_a^b \phi(x)\left\{\frac{d^2}{dx^2}\left\{EI\frac{d^2}{dx^2}Y(x)\right\} + P\left[\frac{d^2}{dx^2}Y(x)\right] - m\,\omega^2\,Y(x)\right\} = 0$$

(11)

$$\int_a^b \phi(x)\left\{\frac{d^2}{dx^2}\left\{EI\frac{d^2}{dx^2}Y(x)\right\}\right\}dx \quad + \int_a^b \phi(x)\left\{P\left[\frac{d^2}{dx^2}Y(x)\right]\right\}dx$$

$$-\int_a^b \phi(x)\left\{m\,\omega^2\,Y(x)\right\}dx \quad = \quad 0$$

(12)

$$\int_a^b \phi(x)\left\{\frac{d^2}{dx^2}\left\{EI\frac{d^2}{dx^2}Y(x)\right\}\right\}dx \quad + \quad P\int_a^b \phi(x)\left\{\frac{d^2}{dx^2}Y(x)\right\}dx$$

$$-m\,\omega^2\int_a^b \phi(x)\left\{Y(x)\right\}dx \quad = \quad 0$$

(13)

The essence of the Galerkin method is that the test function is chosen as

$$\phi(x) = Y(x) \qquad (14)$$

Thus

$$\int_a^b Y(x)\left\{\frac{d^2}{dx^2}\left\{EI\frac{d^2}{dx^2}Y(x)\right\}\right\}dx$$

$$+P\int_a^b Y(x)\left\{\frac{d^2}{dx^2}Y(x)\right\}dx$$

$$-m\,\omega^2\int_a^b Y(x)\{Y(x)\}dx \quad = \quad 0$$

(15)

The first integral in equation (15) yields the following local stiffness matrix per References 2 and 3.

$$K_j = \left(\frac{EI}{h^3}\right)\begin{bmatrix} 12 & 6 & -12 & 6 \\ & 4 & -6 & 2 \\ & & 12 & -6 \\ & & & 4 \end{bmatrix}$$

(16)

The first integral in equation (15) yields the following local mass matrix per References 2 and 3.

$$M_j = \left(\frac{h\,m}{420}\right)\begin{bmatrix} 156 & 22 & 54 & -13 \\ & 4 & 13 & -3 \\ & & 156 & -22 \\ & & & 4 \end{bmatrix}$$

(17)

Evaluate the second term of equation (15).

$$\int_a^b Y(x)\left\{ \frac{d^2}{dx^2} Y(x) \right\} dx =$$

$$\int_a^b \frac{d}{dx}\left\{ Y(x) \frac{d}{dx} Y(x) \right\} dx \quad - \quad \int_a^b \left\{ \frac{d}{dx} Y(x) \frac{d}{dx} Y(x) \right\} dx$$

(18)

$$\int_a^b Y(x)\left\{ \frac{d^2}{dx^2} Y(x) \right\} dx =$$

$$\left\{ Y(x) \frac{d}{dx} Y(x) \right\} \Bigg|_a^b \quad - \quad \int_a^b \left\{ \frac{d}{dx} Y(x) \frac{d}{dx} Y(x) \right\} dx$$

(19)

Note that

$$\frac{d}{dx}\left[EI \frac{d^2 y}{dx^2} \right] + P \frac{dy}{dx} = 0 \quad \text{or} \quad y = 0 \quad \text{at} \quad x = 0 , L$$

(20)

In equation (19), the integration limit a corresponds to zero. The integration limit b corresponds to L.

Thus, the first term on the right side of equation (19) may be omitted. Reference 3, equation (23) is needed to complete the justification.

$$\int_a^b Y(x)\left\{ \frac{d^2}{dx^2} Y(x) \right\}dx \quad = \quad -\int_a^b \left\{ \frac{d}{dx} Y(x) \frac{d}{dx} Y(x) \right\}dx \tag{21}$$

Change the global coordinate x to the local coordinate ξ .

$$\xi = j - x/h \tag{22}$$

$$h\,\xi = h\,j - x \tag{23}$$

$$x = h\,j - h\,\xi \tag{24}$$

$$x = h(j - \xi) \tag{25}$$

$$dx = -h\,d\xi \tag{26}$$

$$\int_a^b Y(x)\left\{ \frac{d^2}{dx^2} Y(x) \right\}dx \quad = h \int_a^b \left\{ \frac{d}{dx} Y(x) \frac{d}{dx} Y(x) \right\}d\xi \tag{27}$$

Now Let

$$Y(x) = \underline{L}^T \quad \bar{a} \,, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{28}$$

$$\bar{a} = \begin{bmatrix} y_{j-1} & h\theta_{j-1} & y_j & h\theta_j \end{bmatrix}^T \tag{29}$$

$$\underline{L} = \begin{bmatrix} 3\,\xi^2 - 2\xi^3 \\ \xi^2 - \xi^3 \\ 1 - 3\,\xi^2 + 2\xi^3 \\ -\xi + 2\xi^2 - \xi^3 \end{bmatrix} \tag{30}$$

$$\underline{L}' = \begin{bmatrix} 6\,\xi - 6\xi^2 \\ 2\xi - 3\xi^2 \\ -6\ \xi + 6\xi^2 \\ -1 + 4\xi - 3\xi^2 \end{bmatrix} \tag{31}$$

$$\underline{L}'' = \begin{bmatrix} 6 - 12\xi \\ 2 - 6\xi \\ -6\ + 12\xi \\ 4 - 6\xi \end{bmatrix} \tag{32}$$

$$\frac{d}{dx} Y(x) = \left(\frac{-1}{h}\right) \ \underline{L'}^{\ T}\ \bar{a}\ , \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{33}$$

$$\int_a^b Y(x) \left\{ \frac{d^2}{dx^2} Y(x) \right\} dx = \frac{1}{h} \int_a^b \left\{ \underline{L'}^{\ T}\ \bar{a}\ \ \underline{L'}^{\ T}\ \bar{a} \right\} d\xi \tag{34}$$

$$\int_a^b Y(x) \left\{ \frac{d^2}{dx^2} Y(x) \right\} dx = \frac{1}{h} \int_a^b \left\{ \bar{a}^T\ \underline{L'}\ \ \underline{L'}^{\ T}\ \bar{a} \right\} d\xi \tag{35}$$

The integral on the right side of equation (35) is evaluated in Appendices A.

Final Assembly of Mass and Stiffness Matrices

$$M_j = \left(\frac{h\,m}{420}\right)\begin{bmatrix} 156 & 22 & 54 & -13 \\ & 4 & 13 & -3 \\ & & 156 & -22 \\ & & & 4 \end{bmatrix} \tag{36}$$

$$K_j = \left(\frac{EI}{h^3}\right)\begin{bmatrix} 12 & 6 & -12 & 6 \\ & 4 & -6 & 2 \\ & & 12 & -6 \\ & & & 4 \end{bmatrix} + \frac{P}{h}\left(\frac{1}{30}\right)\begin{bmatrix} 36 & 3 & -36 & 3 \\ & 4 & -3 & -1 \\ & & 36 & -3 \\ & & & 4 \end{bmatrix} \tag{37}$$

An example is given in Appendix B.

References

1. L. Meirovitch, Analytical Methods in Vibration, Macmillan, New York, 1967.
2. L. Meirovitch, Computational Methods in Structural Dynamics, Sijthoff & Noordhoff, The Netherlands, 1980.
3. T. Irvine, Transverse Vibration of a Beam via the Finite Element Method, Revision A, Vibrationdata, 2000.
4. T. Irvine, Natural Frequencies of Beams Subjected to a Uniform Axial Load, Revision A, Vibrationdata, 2003.

APPENDIX A

Evaluate the following integral:

$$\int_0^1 \left\{ \underline{L}' \, \underline{L}'^{\,T} \right\} d\xi \tag{A-1}$$

$$\underline{L}' \, \underline{L}'^{\,T} = \begin{bmatrix} 6\xi - 6\xi^2 \\ 2\xi - 3\xi^2 \\ -6\xi + 6\xi^2 \\ -1 + 4\xi - 3\xi^2 \end{bmatrix} \begin{bmatrix} 6\xi - 6\xi^2 & 2\xi - 3\xi^2 & -6\xi + 6\xi^2 & -1 + 4\xi - 3\xi^2 \end{bmatrix} \tag{A-2}$$

$\underline{L}' \, \underline{L}'^{\,T} =$

$$\begin{bmatrix} (6\xi - 6\xi^2)^2 & (6\xi - 6\xi^2)(2\xi - 3\xi^2) & (6\xi - 6\xi^2)(-6\xi + 6\xi^2) & (6\xi - 6\xi^2)(-1 + 4\xi - 3\xi^2) \\ & (2\xi - 3\xi^2)^2 & (2\xi - 3\xi^2)(-6\xi + 6\xi^2) & (2\xi - 3\xi^2)(-1 + 4\xi - 3\xi^2) \\ & & (-6\xi + 6\xi^2)^2 & (-6\xi + 6\xi^2)(-1 + 4\xi - 3\xi^2) \\ & & & (-1 + 4\xi - 3\xi^2)^2 \end{bmatrix} \tag{A-3}$$

Integrate each coefficient separately.

$$\int_0^1 \left(6\,\xi - 6\xi^2\right)^2 \, d\xi = \int_0^1 \left(36\xi^2 - 72\xi^3 + 36\xi^4\right) d\xi$$

$$= \left(12\xi^3 - 18\xi^4 + \frac{36}{5}\xi^5\right)\Big|_0^1$$

$$= 12 - 18 + \frac{36}{5}$$

$$= \frac{36}{30} \tag{A-4}$$

$$\int_0^1 \left(6\,\xi - 6\xi^2\right)\left(2\xi - 3\xi^2\right) d\xi = \int_0^1 \left(12\xi^2 - 30\xi^3 + 18\xi^4\right) d\xi$$

$$= \left(4\xi^3 - \frac{15}{2}\xi^4 + \frac{18}{5}\xi^5\right)\Big|_0^1$$

$$= \left(4 - \frac{15}{2} + \frac{18}{5}\right)$$

$$= \frac{3}{30} \tag{A-5}$$

$$\int_0^1 \left(6\xi - 6\xi^2\right)\left(-6\xi + 6\xi^2\right) d\xi = \int_0^1 \left(-36\xi^2 + 72\xi^3 - 36\xi^4\right) d\xi$$

$$= \left(-12\xi^3 + 18\xi^4 - \frac{36}{5}\xi^5\right)\Big|_0^1$$

$$= -12 + 18 - \frac{36}{5}$$

$$= \frac{-36}{30}$$

(A-6)

$$\int_0^1 \left(6\xi - 6\xi^2\right)\left(-1 + 4\xi - 3\xi^2\right) d\xi = \int_0^1 \left(-6\xi + 30\xi^2 - 42\xi^3 + 18\xi^4\right) d\xi$$

$$= \left(-3\xi^2 + 10\xi^3 - \frac{21}{2}\xi^4 + \frac{18}{5}\xi^5\right)\Big|_0^1$$

$$= -3 + 10 - \frac{21}{2} + \frac{18}{5}$$

$$= \frac{3}{30}$$

(A-7)

$$\int_0^1 \left(2\xi - 3\xi^2\right)^2 d\xi = \int_0^1 \left(4\xi^2 - 12\xi^3 + 9\xi^4\right) d\xi$$

$$= \left(\frac{4}{3}\xi^3 - 3\xi^4 + \frac{9}{5}\xi^5\right)\Big|_0^1$$

$$= \frac{4}{3} - 3 + \frac{9}{5}$$

$$= \frac{4}{30}$$

(A-8)

$$\int_0^1 \left(2\xi - 3\xi^2\right)\left(-6\ \xi + 6\xi^2\right) d\xi = \int_0^1 \left(-12\xi^2 + 30\xi^3 - 18\xi^4\right) d\xi$$

$$= \left(-4\xi^3 + \frac{15}{2}\xi^4 - \frac{18}{5}\xi^5\right)\Big|_0^1$$

$$= -4 + \frac{15}{2} - \frac{18}{5}$$

$$= \frac{-3}{30}$$

(A-9)

$$\int_0^1 \left(2\xi - 3\xi^2\right)\left(-1 + 4\xi - 3\xi^2\right)d\xi = \int_0^1 \left(-2\xi + 11\xi^2 - 18\xi^3 + 9\xi^4\right)d\xi$$

$$= \left(-\xi^2 + \frac{11}{3}\xi^3 - \frac{9}{2}\xi^4 + \frac{9}{5}\xi^5\right)\Bigg|_0^1$$

$$= -1 + \frac{11}{3} - \frac{9}{2} + \frac{9}{5}$$

$$= \frac{-1}{30}$$

(A-10)

$$\int_0^1 \left(-6\xi + 6\xi^2\right)^2 d\xi = \int_0^1 \left(36\xi^2 - 72\xi^3 + 36\xi^4\right)d\xi$$

$$= \left(12\xi^3 - 18\xi^4 + \frac{36}{5}\xi^5\right)\Bigg|_0^1$$

$$= 12 - 18 + \frac{36}{5}$$

$$= \frac{36}{30}$$

(A-11)

$$\int_0^1 \left(-6\ \xi + 6\xi^2\right)\left(-1 + 4\xi - 3\xi^2\right)d\xi = \int_0^1 \left(6\xi\ - 30\xi^2\ + 42\xi^3 - 18\xi^4\right)d\xi$$

$$= \left(3\xi^2\ - 10\xi^3\ + \frac{21}{2}\xi^4 - \frac{18}{5}\xi^5\right)\Big|_0^1$$

$$= 3\ - 10\ + \frac{21}{2} - \frac{18}{5}$$

$$= \frac{-3}{30}$$

(A-12)

$$\int_0^1 \left(-1 + 4\xi - 3\xi^2\right)^2\ d\xi = \int_0^1 \left(1 - 8\xi + 22\xi^2 - 24\xi^3 + 9\xi^4\right)d\xi$$

$$= \left(\xi - 4\xi^2 + \frac{22}{3}\xi^3 - 6\xi^4 + \frac{9}{5}\xi^5\right)\Big|_0^1$$

$$= 1 - 4 + \frac{22}{3} - 6 + \frac{9}{5}$$

$$= \frac{4}{30}$$

(A-13)

The assembled matrix is

$$\left(\frac{1}{30}\right)\begin{bmatrix} 36 & 3 & -36 & 3 \\ & 4 & -3 & -1 \\ & & 36 & -3 \\ & & & 4 \end{bmatrix}$$

(A-14)

APPENDIX B

Example 1

Model the cantilever beam in Figure 1 as a single element using the mass and stiffness matrices in equations 36 and 37. The model consists of one element and two nodes as shown in Figure B-1.

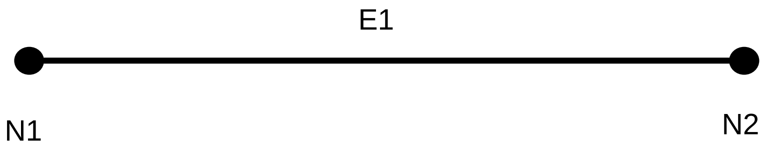

Figure B-1.

Note that h = L.

The mass matrix is

$$\underline{M} = \left(\frac{L\,m}{420}\right)\begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix} \tag{B-1}$$

The stiffness matrix is

$$K_j = \left(\frac{EI}{h^3}\right)\begin{bmatrix} 12 & 6 & -12 & 6 \\ & 4 & -6 & 2 \\ & & 12 & -6 \\ & & & 4 \end{bmatrix}$$

$$+\frac{P}{h}\left(\frac{1}{30}\right)\begin{bmatrix} 36 & 3 & -36 & 3 \\ & 4 & -3 & -1 \\ & & 36 & -3 \\ & & & 4 \end{bmatrix} \tag{B-2}$$

The buckling load for a fixed-free beam is

$$P_{cr} = \frac{\pi^2 EI}{4L^2} \tag{B-3}$$

Let

$$P = 0.4\, P_{cr} \tag{B-4}$$

$$P = \frac{0.4\, \pi^2 EI}{4L^2} \tag{B-5}$$

$$P = \frac{\pi^2 EI}{10L^2} \tag{B-6}$$

$$K_j = \left(\frac{EI}{L^3}\right)\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

$$+ \frac{\pi^2 EI}{10L^2}\left(\frac{1}{L}\right)\left(\frac{1}{30}\right)\begin{bmatrix} 36 & 3 & -36 & 3 \\ & 4 & -3 & -1 \\ & & 36 & -3 \\ & & & 4 \end{bmatrix} \tag{B-7}$$

$$K_j = \left(\frac{EI}{L^3}\right)\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

$$+\frac{\pi^2}{300}\left(\frac{EI}{L^3}\right)\begin{bmatrix} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & -1 \\ -36 & -3 & 36 & -3 \\ 3 & -1 & -3 & 4 \end{bmatrix}$$

(B-8)

$$K_j = \left(\frac{EI}{L^3}\right)\begin{bmatrix} 13.18 & 6.099 & -13.18 & 6.099 \\ 6.099 & 4.131 & -6.099 & 1.967 \\ -13.18 & -6.099 & 13.18 & -6.099 \\ 6.099 & 1.967 & -6.099 & 4.131 \end{bmatrix}$$

(B-9)

$$\left(\frac{EI}{L^3}\right)\begin{bmatrix} 13.18 & 6.099 & -13.18 & 6.099 \\ 6.099 & 4.131 & -6.099 & 1.967 \\ -13.18 & -6.099 & 13.18 & -6.099 \\ 6.099 & 1.967 & -6.099 & 4.131 \end{bmatrix}\begin{bmatrix} y_1 \\ h\,\theta_1 \\ y_2 \\ h\,\theta_2 \end{bmatrix} = \omega^2\left(\frac{L\,m}{420}\right)\begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ h\,\theta_1 \\ y_2 \\ h\,\theta_2 \end{bmatrix}$$

(B-10)

$$\begin{bmatrix} 13.18 & 6.099 & -13.18 & 6.099 \\ 6.099 & 4.131 & -6.099 & 1.967 \\ -13.18 & -6.099 & 13.18 & -6.099 \\ 6.099 & 1.967 & -6.099 & 4.131 \end{bmatrix}\begin{bmatrix} y_1 \\ h\,\theta_1 \\ y_2 \\ h\,\theta_2 \end{bmatrix} = \lambda\begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ h\,\theta_1 \\ y_2 \\ h\,\theta_2 \end{bmatrix}$$

(B-11)

where

$$\lambda = \left(\frac{L^4 m}{420\ EI} \right) \omega^2 \qquad \text{(B-12)}$$

$$\omega = \left[\sqrt{\frac{420\ EI}{L^4 m}} \right] \sqrt{\lambda} \qquad \text{(B-13)}$$

The boundary conditions at node 1 are

$$y_1 = 0 \qquad \text{(B-14)}$$

$$\theta_1 = 0 \qquad \text{(B-15)}$$

The first two columns and the first two rows of each matrix can thus be struck out.

The resulting eigen equation is thus

$$\begin{bmatrix} 13.18 & 6.099 & -13.18 & 6.099 \\ 6.099 & 4.131 & -6.099 & 1.967 \\ -13.18 & -6.099 & 13.18 & -6.099 \\ 6.099 & 1.967 & -6.099 & 4.131 \end{bmatrix} \begin{bmatrix} y_1 \\ h\,\theta_1 \\ y_2 \\ h\,\theta_2 \end{bmatrix} = \lambda \begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ h\,\theta_1 \\ y_2 \\ h\,\theta_2 \end{bmatrix}$$

(B-16)

$$\begin{bmatrix} 13.18 & -6.099 \\ -6.099 & 4.131 \end{bmatrix} \begin{bmatrix} y_2 \\ h\,\theta_2 \end{bmatrix} = \lambda \begin{bmatrix} 156 & -22 \\ -22 & 4 \end{bmatrix} \begin{bmatrix} y_2 \\ h\,\theta_2 \end{bmatrix}$$

(B-17)

The eigenvalues are found using the method in Reference 2.

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0.04077 \\ 3.022 \end{bmatrix} \qquad \text{(B-18)}$$

$$\begin{bmatrix} \sqrt{\lambda_1} \\ \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} 0.202 \\ 1.738 \end{bmatrix} \tag{B-19}$$

The finite element results for the natural frequencies are thus

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \sqrt{\frac{420\ EI}{m L^4}} \begin{bmatrix} 0.202 \\ 1.738 \end{bmatrix} \tag{B-20}$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \sqrt{\frac{EI}{m L^4}} \begin{bmatrix} 4.138 \\ 35.626 \end{bmatrix} \tag{B-21}$$

The finite element results are compared to the classical results in Table B-1.

Table B-1.

P = 0.4 Pcr Case, Natural Frequency Comparison, 1 Element

Mode	Finite Element Model $\omega \sqrt{\frac{m L^4}{EI}}$	Classical Solution $\omega \sqrt{\frac{m L^4}{EI}}$
1	4.132	4.160

The finite element value is 0.67% lower than the classical solution. The classical result is taken from Reference 4.

Note that $\omega \sqrt{\frac{\rho L^4}{EI}} = 3.5160$ for the case where P = 0, per the classical solution in Reference 3.

The analysis is repeated for other model sizes, representing the same beam, in Table B-2.

Table B-2.

P = 0.4 Pcr Case, Fundamental Frequency for Various Model Sizes

Elements in Model	Finite Element Model $\omega\sqrt{\frac{mL^4}{EI}}$	Classical Solution $\omega\sqrt{\frac{mL^4}{EI}}$	Error
1	4.132	4.160	0.67 %
2	4.117	4.160	1.03 %
4	4.103	4.160	1.37 %
8	4.103	4.160	1.37 %
16	4.103	4.160	1.37 %