NATURAL FREQUENCIES OF BEAMS SUBJECTED TO A UNIFORM AXIAL LOAD

Revision C

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The natural frequency of a beam is increased by an axial tension load and decreased by an axial compressive load.

The governing differential equation for the transverse displacement y(x, t) is

$$\frac{\partial^2}{\partial x^2}\left\{EI(x)\frac{\partial^2}{\partial x^2}y(x,t)\right\}-\frac{\partial}{\partial x}\left[P(x)\frac{\partial}{\partial x}y(x,t)\right]+m\frac{\partial^2 y(x,t)}{\partial t^2}=0 \tag{1a}$$

where

E	=	Elastic modulus
I	=	Area moment of inertia
m	=	Mass per length
L	=	Length
P	=	Axial tension load

Equation (1) is taken from Reference 1, equation (10.24) for the case of zero transverse loading.

For a constant axial load,

$$\frac{\partial^2}{\partial x^2}\left\{EI(x)\frac{\partial^2}{\partial x^2}y(x,t)\right\}-P\frac{\partial^2}{\partial x^2}y(x,t)+m\frac{\partial^2 y(x,t)}{\partial t^2}=0 \tag{1b}$$

Natural Frequency Formulas

Natural frequency formulas are given in References 2 through 4.

The exact natural frequency f_n for a pinned-pinned or sliding-sliding beam is

$$f_n = \frac{n^2\pi^2}{2\pi L^2}\sqrt{1+\frac{PL^2}{EI\,n^2\pi^2}}\sqrt{\frac{EI}{m}} \quad , \; n=1, 2, 3, \ldots \tag{2}$$

Note that P is positive for a tension load. P is negative for a compression load.

The exact natural frequency for a sliding-pinned beam is

$$f_n = \frac{n^2\pi^2}{8\pi L^2}\sqrt{1+\frac{4PL^2}{EI\,n^2\pi^2}}\sqrt{\frac{EI}{m}} \quad , \; n=1, 3, 5, \ldots \tag{3}$$

The approximate natural frequency formula for beams with other boundary conditions is

$$\frac{f_n\big|_{P\neq 0}}{f_n\big|_{P=0}} = \sqrt{1+\frac{P}{\left|P_{cr}\right|}\frac{\lambda_1^2}{\lambda_n^2}} \quad , \; n=1, 2, 3, \ldots \tag{4}$$

P_{cr} is the buckling load, as given in Appendix A for common boundary conditions.

λ_n is the non-dimensional frequency in the absence of an axial load. Values for common boundary conditions are given in Appendix B.

Note that the fundamental frequency approaches zero as P approaches the negative critical load. Again, the negative sign corresponds to compression.

An example is shown in Appendix C.

References

1. L. Meirovitch, Analytical Methods in Vibration, Macmillan, New York, 1967.
2. R. Blevins, Formulas for Natural Frequency and Mode Shape, Krieger, Malabar, Florida, 1979.
3. Shaker, F.J., “Effect of Axial Load on Mode Shapes and Frequencies of Beams,” Lewis Research Center Report NASA-TN-8109, December 1975.
4. C. Harris, editor; Shock and Vibration Handbook, 4th edition; W. Stokey, "Vibration of Systems Having Distributed Mass and Elasticity," McGraw-Hill, New York, 1988.
5. Timoshenko and Gere, Theory of Elastic Stability, International Student Edition, 2nd Edition, McGraw-Hill, New Delhi, 1963.
6. Alexander Chajes, Principles of Structural Stability Theory, Prentice-Hall, New Jersey, 1974.
7. T. Irvine, Application of the Newton-Raphson Method to Vibration Problems, Revision B, Vibrationdata Publications, 1999.
8. T. Irvine, Bending Frequencies of Beams, Rods, and Pipes, Revision H; Vibrationdata Publications, 2002.

APPENDIX A

Critical Buckling Loads for Beams with a Constant Axial Load

Boundary Condition	P_{cr}
Fixed-Fixed	$\frac{4\pi^2 EI}{L^2}$
Fixed-Pinned	$\frac{2.046\pi^2 EI}{L^2}$
Fixed-Free	$\frac{\pi^2 EI}{4L^2}$
Pinned-Pinned or Free-Free	$\frac{\pi^2 EI}{L^2}$

The critical loads are taken from References 5 and 6.

APPENDIX B

Non-dimensional Frequency Parameters

The values in the following tables are taken from Reference 7.

Table B-1. Fixed-Free	
n	λ_n
1	1.875104
2	4.694091

Table B-2. Free-Free or Fixed-Fixed	
n	λ_n
1	4.73004
2	7.85320

Table B-3. Free-Pinned or Fixed-Pinned	
n	λ_n
1	3.926602
2	7.068583

APPENDIX C

Example

Consider a fixed-free beam made from aluminum. The beam is 24 inches long. It has a circular cross-section with a diameter of 1 inch.

It is subjected to an axial load of +833 lbf, where the positive sign indicates tension. Calculate the fundamental frequency for the loaded beam.

E	=	9900000. lbf/in^2	(elastic modulus)
I	=	0.0490874 in^4	(area moment inertia)
ρ	=	0.098 lbm/in^3 = 0.00025386 lbf sec^2/in^4	(mass/volume)
L	=	24 inch	(length)
A	=	0.7854 in^2	(cross-section area)
P	=	833 lbf	(axial load)

$$m = \rho \, A \tag{C-1}$$

$$m = (0.00025386 \text{ lbf sec^2/in^4})(0.7854 \text{ in^2}) = 1.99\text{e-04 lbf sec^2/in^2} \tag{C-2}$$

The fundamental frequency for the beam with no axial load is

$$f_1 = \frac{1}{2\pi}\left[\frac{3.5156}{L^2}\right]\sqrt{\frac{EI}{m}} \tag{C-3}$$

$$f_1 = 47.8 \text{ Hz} \quad (\text{ for } P = 0) \tag{C-4}$$

Equation (C-3) is taken from Reference 8.

The critical buckling load is

$$P_{cr} = \frac{\pi^2 \, EI}{4L^2} \tag{C-5}$$

$$P_{cr} = 2082 \text{ lbf} \tag{C-6}$$

$$\frac{f_n \big|_{P \neq 0}}{f_n \big|_{P=0}} = \sqrt{1 + \frac{P}{\left| P_{cr} \right|} \frac{\lambda_1^2}{\lambda_n^2}} \quad , \; n=1, 2, 3, \ldots \tag{C-7}$$

The fundamental frequency requires n = 1.

$$\frac{f_n \big|_{P \neq 0}}{f_n \big|_{P=0}} = \sqrt{1 + \frac{P}{\left| P_{cr} \right|}} \tag{C-8}$$

$$\frac{f_n \big|_{P \neq 0}}{f_n \big|_{P=0}} = \sqrt{1 + \frac{833}{\left| 2082 \right|}} \tag{C-9}$$

$$\frac{f_n \big|_{P \neq 0}}{f_n \big|_{P=0}} = 1.18 \tag{C-10}$$

$$f_n \big|_{P \neq 0} = 1.18\,(47.8 \text{ Hz}) \tag{C-11}$$

$$f_n \big|_{P \neq 0} = 56.6 \text{ Hz} \tag{C-12}$$

The result was verified using a finite element model. The software was NX/Nastran. The finite element frequency was 55.89 Hz, using a model with 24 elements. The mode shape is shown in Figure C-1.

The NX/Nastran input file is shown in Appendix D.

An alternate NEi/Nastran file is shown in Appendix E.

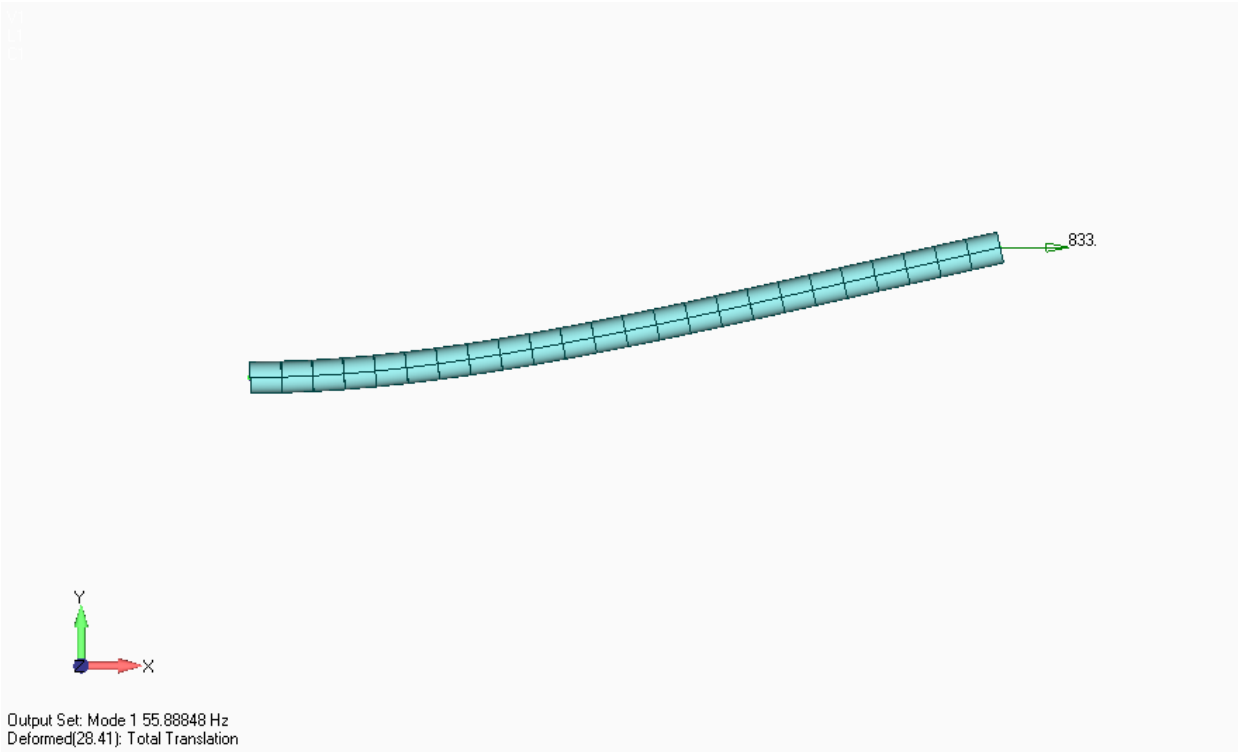

Figure C-1. Fundamental Mode Shape, Cantilever Beam with Axial Load

APPENDIX D

NX/Nastran File

```
ID C:\NENASTRAN82\Modeler\ca,NE/Na
SOL 103
TIME 10000
CEND
TITLE = Normal Modes
ECHO = NONE
DISPLACEMENT(PRINT) = ALL
SUBCASE 1
LABEL = PRESTRESS TENSILE LOAD (AXIAL)
SPC = 1
LOAD = 1
SUBCASE 2
LABEL = MODAL
SPC = 1
METHOD = 1
STATSUB = 1
BEGIN BULK
$ ***************************************************************************
$   Written by : Femap with NX Nastran
$   Version    : 10.10
$   Translator : NEiNastran
$   From Model : C:\vibrationdata\tutorials\beam_axial_833.MOD
$   Date       : Fri Jul 30 09:40:37 2010
$ ***************************************************************************
$
PARAM,OGEOM,NO
PARAM,AUTOSPC,YES
PARAM,GRDPNT,0
EIGRL          1                      10                              MASS
CORD2C         1       0      0.      0.      0.      0.      0.
1.+FEMAPC1
+FEMAPC1      1.      0.      1.
CORD2S         2       0      0.      0.      0.      0.      0.
1.+FEMAPC2
+FEMAPC2      1.      0.      1.
$ Femap with NX Nastran Load Set 1 : set 1
FORCE          1      25       0      1.    833.      0.      0.
$ Femap with NX Nastran Constraint Set 1 : set 1
SPC            1       1  123456      0.
SPC            1       2     345      0.
SPC            1       3     345      0.
SPC            1       4     345      0.
SPC            1       5     345      0.
SPC            1       6     345      0.
SPC            1       7     345      0.
SPC            1       8     345      0.
SPC            1       9     345      0.
SPC            1      10     345      0.
SPC            1      11     345      0.
SPC            1      12     345      0.
```

```
SPC            1      13     345      0.
SPC            1      14     345      0.
SPC            1      15     345      0.
SPC            1      16     345      0.
SPC            1      17     345      0.
SPC            1      18     345      0.
SPC            1      19     345      0.
SPC            1      20     345      0.
SPC            1      21     345      0.
SPC            1      22     345      0.
SPC            1      23     345      0.
SPC            1      24     345      0.
SPC            1      25     345      0.
$ Femap with NX Nastran Property 1 : Bar
$ Femap with NX Nastran PropShape 1 : 5,0,0.5,0.,0.,0.,0.,0.
$ Femap with NX Nastran PropOrient 1 : 5,0,0.,1.,2.,3.,4.,-1.,0.,0.
PBAR           1       1 .785398.0490874.0490874.0980921      0.        +PR
1
+PR    1      0.     -.5      .5      0.      0.      .5     -.5      0.+PA
1
+PA    1.8861609.8861584      0.
$ Femap with NX Nastran Material 1 : 6061-T651 Al Plate .25-2.
MAT1           19900000.3721805.     .33 2.539-4 1.265-5     70.        +MT
1
+MT    1  35000.  35000.  27000.
MAT4           1  .00206  81.144 2.539-4                         1.
GRID           1       0      0.      0.      0.       0
GRID           2       0      1.      0.      0.       0
GRID           3       0      2.      0.      0.       0
GRID           4       0      3.      0.      0.       0
GRID           5       0      4.      0.      0.       0
GRID           6       0      5.      0.      0.       0
GRID           7       0      6.      0.      0.       0
GRID           8       0      7.      0.      0.       0
GRID           9       0      8.      0.      0.       0
GRID          10       0      9.      0.      0.       0
GRID          11       0     10.      0.      0.       0
GRID          12       0     11.      0.      0.       0
GRID          13       0     12.      0.      0.       0
GRID          14       0     13.      0.      0.       0
GRID          15       0     14.      0.      0.       0
GRID          16       0     15.      0.      0.       0
GRID          17       0     16.      0.      0.       0
GRID          18       0     17.      0.      0.       0
GRID          19       0     18.      0.      0.       0
GRID          20       0     19.      0.      0.       0
GRID          21       0     20.      0.      0.       0
GRID          22       0     21.      0.      0.       0
GRID          23       0     22.      0.      0.       0
GRID          24       0     23.      0.      0.       0
GRID          25       0     24.      0.      0.       0
CBAR           1       1       1       2      0.      1.      0.
CBAR           2       1       2       3      0.      1.      0.
CBAR           3       1       3       4      0.      1.      0.
CBAR           4       1       4       5      0.      1.      0.
CBAR           5       1       5       6      0.      1.      0.
CBAR           6       1       6       7      0.      1.      0.
```

```
CBAR           7       1       7       8      0.      1.      0.
CBAR           8       1       8       9      0.      1.      0.
CBAR           9       1       9      10      0.      1.      0.
CBAR          10       1      10      11      0.      1.      0.
CBAR          11       1      11      12      0.      1.      0.
CBAR          12       1      12      13      0.      1.      0.
CBAR          13       1      13      14      0.      1.      0.
CBAR          14       1      14      15      0.      1.      0.
CBAR          15       1      15      16      0.      1.      0.
CBAR          16       1      16      17      0.      1.      0.
CBAR          17       1      17      18      0.      1.      0.
CBAR          18       1      18      19      0.      1.      0.
CBAR          19       1      19      20      0.      1.      0.
CBAR          20       1      20      21      0.      1.      0.
CBAR          21       1      21      22      0.      1.      0.
CBAR          22       1      22      23      0.      1.      0.
CBAR          23       1      23      24      0.      1.      0.
CBAR          24       1      24      25      0.      1.      0.
ENDDATA
```

APPENDIX E

NEI/Nastran File

```
ID C:\NENASTRAN82\Modeler\ca,NE/Na
SOL LINEAR PRESTRESS MODAL
TIME 10000
CEND
TITLE = Normal Modes
ECHO = NONE
DISPLACEMENT(PRINT) = ALL
SUBCASE 1
LABEL = PRESTRESS TENSILE LOAD (AXIAL)
SPC = 1
LOAD = 1
SUBCASE 2
LABEL = MODAL
SPC = 1
METHOD = 1
BEGIN BULK
$
******************************************************************
*********
$ Written by : NE/Nastran for Windows
$ Version : 8.20
$ Translator : NE/Nastran
$ From Model : C:\NENASTRAN82\Modeler\cantilever_beam3.MOD
$ Date : Mon Jul 21 12:05:50 2003
$
******************************************************************
*********
$
PARAM,POST,-1
PARAM,OGEOM,NO
PARAM,AUTOSPC,YES
PARAM,GRDPNT,0
EIGRL 1 10
MASS
CORD2C 1 0 0. 0. 0. 0. 0.
1.+NE/NAC1
+NE/NAC1 1. 0. 1.
CORD2S 2 0 0. 0. 0. 0. 0.
1.+NE/NAC2
+NE/NAC2 1. 0. 1.
$ NE/Nastran for Windows Load Set 1 : set 1
FORCE 1 25 0 1. 833. 0. 0.
$ NE/Nastran for Windows Constraint Set 1 : set 1
SPC 1 1 123456 0.
$ NE/Nastran for Windows Property 1 : Bar
PBAR 1 1 0.785070.0490870.0490870.098092 0.
+PR 1
+PR 1 0. -0.5 0.5 0. 0. 0.5 -0.5
0.+PA 1
+PA 1 0.88652 0.88653 0.
$ NE/Nastran for Windows Material 1 : 6061-T651 Al Plate .25-2.
```

```
MAT1 19900000. 0.332.539E-41.265E-5 70.
+MT 1
+MT 1 35000. 35000. 27000.
10
MAT4 12.060E-3 81.1442.539E-4
GRID 1 0 0. 0. 0. 0
GRID 2 0 1. 0. 0. 0
GRID 3 0 2. 0. 0. 0
GRID 4 0 3. 0. 0. 0
GRID 5 0 4. 0. 0. 0
GRID 6 0 5. 0. 0. 0
GRID 7 0 6. 0. 0. 0
GRID 8 0 7. 0. 0. 0
GRID 9 0 8. 0. 0. 0
GRID 10 0 9. 0. 0. 0
GRID 11 0 10. 0. 0. 0
GRID 12 0 11. 0. 0. 0
GRID 13 0 12. 0. 0. 0
GRID 14 0 13. 0. 0. 0
GRID 15 0 14. 0. 0. 0
GRID 16 0 15. 0. 0. 0
GRID 17 0 16. 0. 0. 0
GRID 18 0 17. 0. 0. 0
GRID 19 0 18. 0. 0. 0
GRID 20 0 19. 0. 0. 0
GRID 21 0 20. 0. 0. 0
GRID 22 0 21. 0. 0. 0
GRID 23 0 22. 0. 0. 0
GRID 24 0 23. 0. 0. 0
GRID 25 0 24. 0. 0. 0
CBAR 1 1 1 2 0. 1. 0.
CBAR 2 1 2 3 0. 1. 0.
CBAR 3 1 3 4 0. 1. 0.
CBAR 4 1 4 5 0. 1. 0.
CBAR 5 1 5 6 0. 1. 0.
CBAR 6 1 6 7 0. 1. 0.
CBAR 7 1 7 8 0. 1. 0.
CBAR 8 1 8 9 0. 1. 0.
CBAR 9 1 9 10 0. 1. 0.
CBAR 10 1 10 11 0. 1. 0.
CBAR 11 1 11 12 0. 1. 0.
CBAR 12 1 12 13 0. 1. 0.
CBAR 13 1 13 14 0. 1. 0.
CBAR 14 1 14 15 0. 1. 0.
CBAR 15 1 15 16 0. 1. 0.
CBAR 16 1 16 17 0. 1. 0.
CBAR 17 1 17 18 0. 1. 0.
CBAR 18 1 18 19 0. 1. 0.
CBAR 19 1 19 20 0. 1. 0.
CBAR 20 1 20 21 0. 1. 0.
CBAR 21 1 21 22 0. 1. 0.
CBAR 22 1 22 23 0. 1. 0.
CBAR 23 1 23 24 0. 1. 0.
CBAR 24 1 24 25 0. 1. 0.
ENDDATA
```