

BENDING FREQUENCIES OF BEAMS, RODS, AND PIPES

Revision A

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Introduction

Consider a free-free beam that is mounted on an elastic foundation.

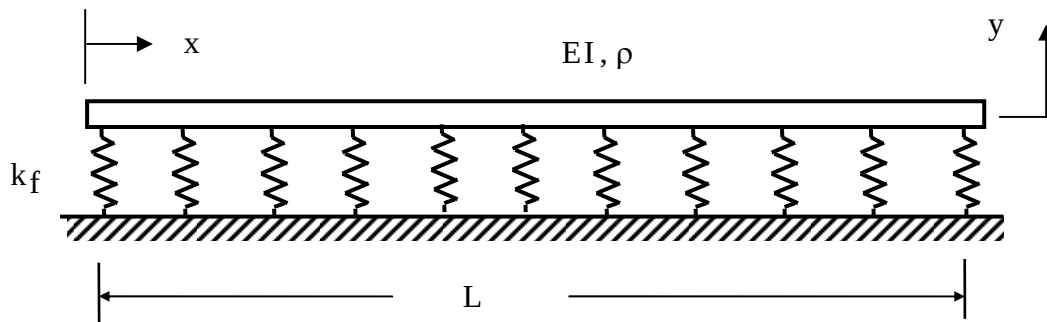


Figure 1.

The governing differential equation is

$$EI \frac{\partial^4 y}{\partial x^4} + k_f y = -\rho \frac{\partial^2 y}{\partial t^2} \quad (1)$$

where

- E is the modulus of elasticity
- I is the area moment of inertia
- L is the length
- ρ is the mass density (mass/length)
- k_f is the foundation stiffness per length (force/length²)

Note that this equation neglects shear deformation and rotary inertia.

Separate the dependent variable.

$$y(x, t) = Y(x)T(t) \quad (2)$$

$$EI \frac{\partial^4 [Y(x)T(t)]}{\partial x^4} + k_f Y(x)T(t) = -\rho \frac{\partial^2 [Y(x)T(t)]}{\partial t^2} \quad (3)$$

$$T(t) \left\{ EI \frac{d^4}{dx^4} Y(x) + k_f Y(x) \right\} = -\rho Y(x) \left\{ \frac{d^2}{dt^2} T(t) \right\} \quad (4)$$

$$\left\{ \frac{-1}{\rho} \right\} \frac{\left\{ EI \frac{d^4}{dx^4} Y(x) + k_f Y(x) \right\}}{Y(x)} = \frac{\left\{ \frac{d^2}{dt^2} T(t) \right\}}{T(t)} \quad (5)$$

Let c be a constant

$$\left\{ \frac{-1}{\rho} \right\} \frac{\left\{ EI \frac{d^4}{dx^4} Y(x) + k_f Y(x) \right\}}{Y(x)} = \frac{\left\{ \frac{d^2}{dt^2} T(t) \right\}}{T(t)} = -c^2 \quad (6)$$

Separate the time variable.

$$\frac{\left\{ \frac{d^2}{dt^2} T(t) \right\}}{T(t)} = -c^2 \quad (7)$$

$$\frac{d^2}{dt^2} T(t) + c^2 T(t) = 0 \quad (8)$$

Propose a solution

$$T(t) = a \sin(\omega t) + b \cos(\omega t) \quad (9)$$

$$\frac{d}{dt} T(t) = a \omega \cos(\omega t) - b \omega \sin(\omega t) \quad (10)$$

$$\frac{d^2}{dt^2} T(t) = -a \omega^2 \sin(\omega t) - b \omega^2 \cos(\omega t) \quad (11)$$

$$-a \omega^2 \sin(\omega t) - b \omega^2 \cos(\omega t) + c^2 [a \sin(\omega t) + b \cos(\omega t)] = 0 \quad (12)$$

$$c = \omega \quad (13)$$

Separate the spatial variable.

$$\left\{ \frac{-1}{\rho} \right\} \frac{\left\{ EI \frac{d^4}{dx^4} Y(x) + k_f Y(x) \right\}}{Y(x)} = -\omega^2 \quad (14)$$

$$EI \frac{d^4}{dx^4} Y(x) + k_f Y(x) - \rho \omega^2 Y(x) = 0 \quad (15)$$

$$\frac{d^4}{dx^4} Y(x) + \left[\frac{1}{EI} \right] \left[k_f - \rho \omega^2 \right] Y(x) = 0 \quad (16)$$

A solution for equation (16) is

$$Y(x) = a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x) \quad (17)$$

$$\frac{dY(x)}{dx} = a_1 \beta \cosh(\beta x) + a_2 \beta \sinh(\beta x) + a_3 \beta \cos(\beta x) - a_4 \beta \sin(\beta x) \quad (18)$$

$$\frac{d^2 Y(x)}{dx^2} = a_1 \beta^2 \sinh(\beta x) + a_2 \beta^2 \cosh(\beta x) - a_3 \beta^2 \sin(\beta x) - a_4 \beta^2 \cos(\beta x) \quad (19)$$

$$\frac{d^3 Y(x)}{dx^3} = a_1 \beta^3 \cosh(\beta x) + a_2 \beta^3 \sinh(\beta x) - a_3 \beta^3 \cos(\beta x) + a_4 \beta^3 \sin(\beta x) \quad (20)$$

$$\frac{d^4 Y(x)}{dx^4} = a_1 \beta^4 \sinh(\beta x) + a_2 \beta^4 \cosh(\beta x) + a_3 \beta^4 \sin(\beta x) + a_4 \beta^4 \cos(\beta x) \quad (21)$$

By substitution,

$$\begin{aligned} & \left\{ a_1 \beta^4 \sinh(\beta x) + a_2 \beta^4 \cosh(\beta x) + a_3 \beta^4 \sin(\beta x) + a_4 \beta^4 \cos(\beta x) \right\} \\ & + \left[\frac{1}{EI} \right] \left[k_f - \rho \omega^2 \right] \{ a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x) \} = 0 \end{aligned} \quad (22)$$

$$\begin{aligned}
& \beta^4 \{a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x)\} \\
& + \left[\frac{1}{EI} \right] \left[k_f - \rho \omega^2 \right] \{a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x)\} = 0
\end{aligned}
\tag{23}$$

The equation is satisfied if

$$\beta^4 = \left[\frac{1}{EI} \right] \left[-k_f + \rho \omega^2 \right]
\tag{24}$$

$$\beta = \left\{ \left[\frac{1}{EI} \right] \left[-k_f + \rho \omega^2 \right] \right\}^{1/4}
\tag{25}$$

Apply the boundary conditions.

$$\left. \frac{d^2 Y}{dx^2} \right|_{x=0} = 0 \quad (\text{zero bending moment})
\tag{26}$$

$$a_2 - a_4 = 0
\tag{27}$$

$$a_4 = a_2
\tag{28}$$

$$\left. \frac{d^3 Y}{dx^3} \right|_{x=0} = 0 \quad (\text{zero shear force})
\tag{29}$$

$$a_1 - a_3 = 0
\tag{30}$$

$$a_3 = a_1
\tag{31}$$

$$\frac{d^2 Y(x)}{dx^2} = a_1 \beta^2 [\sinh(\beta x) - \sin(\beta x)] + a_2 \beta^2 [\cosh(\beta x) - \cos(\beta x)] \quad (32)$$

$$\frac{d^3 Y(x)}{dx^3} = a_1 \beta^3 [\cosh(\beta x) - \cos(\beta x)] + a_2 \beta^3 [\sinh(\beta x) + \sin(\beta x)] \quad (33)$$

$$\left. \frac{d^2 Y}{dx^2} \right|_{x=L} = 0 \quad (\text{zero bending moment}) \quad (34)$$

$$a_1 [\sinh(\beta L) - \sin(\beta L)] + a_2 [\cosh(\beta L) - \cos(\beta L)] = 0 \quad (35)$$

$$\left. \frac{d^3 Y}{dx^3} \right|_{x=L} = 0 \quad (\text{zero shear force}) \quad (36)$$

$$a_1 [\cosh(\beta L) - \cos(\beta L)] + a_2 [\sinh(\beta L) + \sin(\beta L)] = 0 \quad (37)$$

Equation (35) and (37) can be arranged in matrix form.

$$\begin{bmatrix} \sinh(\beta L) - \sin(\beta L) & \cosh(\beta L) - \cos(\beta L) \\ \cosh(\beta L) - \cos(\beta L) & \sinh(\beta L) + \sin(\beta L) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (38)$$

Set the determinant equal to zero.

$$[\sinh(\beta L) - \sin(\beta L)][\sinh(\beta L) + \sin(\beta L)] - [\cosh(\beta L) - \cos(\beta L)]^2 = 0 \quad (39)$$

$$\sinh^2(\beta L) - \sin^2(\beta L) - \cosh^2(\beta L) + 2 \cosh(\beta L) \cos(\beta L) - \cos^2(\beta L) = 0 \quad (40)$$

$$+ 2 \cosh(\beta L) \cos(\beta L) - 2 = 0 \quad (41)$$

$$\cosh(\beta L) \cos(\beta L) - 1 = 0 \quad (42)$$

The roots can be found via the Newton-Raphson method, Reference 2. The first root is

$$\beta L = 4.73004 \quad (43)$$

$$\beta_n = \left\{ \left[\frac{1}{EI} \right] \left[-k_f + \rho \omega_n^2 \right] \right\}^{1/4} \quad (44)$$

$$\beta_n^4 = \left[\frac{1}{EI} \right] \left[-k_f + \rho \omega_n^2 \right] \quad (45)$$

$$\left[\frac{1}{EI} \right] \left[-k_f + \rho \omega_n^2 \right] = \beta_n^4 \quad (46)$$

$$\omega_n^2 = \frac{1}{\rho} \left[EI \beta_n^4 + k_f \right] \quad (47)$$

$$\omega_n = \sqrt{\frac{1}{\rho} \left[EI \beta_n^4 + k_f \right]} \quad (48)$$

The fundamental bending frequency is

$$\omega_1 = \sqrt{\frac{1}{\rho} \left[EI \left[\frac{4.73004}{L} \right]^4 + k_f \right]} \quad (49)$$

Again,

- ρ is the mass density (mass/length)
 k_f is the foundation stiffness per length (force/length²)

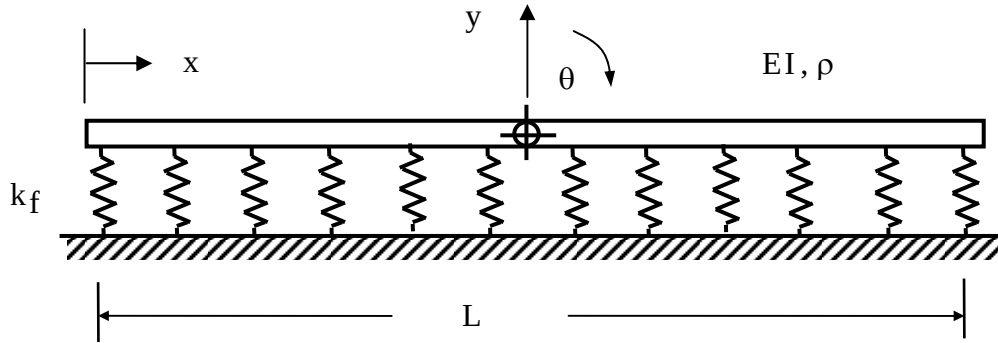
In addition, there are two modes whereby the beam behaves as a rigid body as shown in Appendix A.

References

1. Weaver, Timoshenko, and Young; Vibration Problems in Engineering, Wiley-Interscience, New York, 1990.
2. T. Irvine, Application of the Newton-Raphson Method to Vibration Problems, Vibrationdata Publications, Rev C, 2003.

APPENDIX A

Consider the beam as behaving as a rigid mass.



Assume that the CG is at: $x = L/2$.

Again,

ρ is the mass density (mass/length)

k_f is the foundation stiffness per length (force/length²)

The kinetic energy is

$$T = \frac{1}{2} \rho L \dot{y}^2 + \frac{1}{2} J \dot{\theta}^2 \quad (\text{A-1})$$

The potential energy is

$$V = \frac{1}{2} k_f \left\{ \int_0^L \left[y + \frac{L}{2} \theta - \theta x \right]^2 dx \right\} \quad (\text{A-2})$$

$$V = \frac{1}{2}k_f \int_0^L \left[\left(y + \frac{L}{2}\theta \right)^2 - 2\theta x \left(y + \frac{L}{2}\theta \right) + \theta^2 x^2 \right] dx \quad (A-3)$$

$$V = \frac{1}{2}k_f \left\{ \left(y + \frac{L}{2}\theta \right)^2 x - \theta x^2 \left(y + \frac{L}{2}\theta \right) + \frac{1}{3}\theta^2 x^3 \right\} \Bigg|_0^L \quad (A-4)$$

$$V = \frac{1}{2}k_f \left\{ \left(y + \frac{L}{2}\theta \right)^2 L - \theta L^2 \left(y + \frac{L}{2}\theta \right) + \frac{1}{3}\theta^2 L^3 \right\} \quad (A-5)$$

$$V = \frac{1}{2}k_f L \left\{ \left(y + \frac{L}{2}\theta \right)^2 - \theta L \left(y + \frac{L}{2}\theta \right) + \frac{1}{3}\theta^2 L^2 \right\} \quad (A-6)$$

$$V = \frac{1}{2}k_f L \left\{ y^2 + yL\theta + \frac{L^2}{4}\theta^2 - \theta Ly - \frac{L^2}{2}\theta^2 + \frac{1}{3}\theta^2 L^2 \right\} \quad (A-7)$$

$$V = \frac{1}{2}k_f L \left\{ y^2 + \frac{1}{12} L^2 \theta^2 \right\} \quad (A-8)$$

The total energy of a conservative system is constant. Thus,

$$\frac{d}{dt}(T + V) = 0 \quad (A-9)$$

$$\frac{d}{dt} \left(\frac{1}{2} \rho L \dot{y}^2 + \frac{1}{2} J \dot{\theta}^2 + \frac{1}{2} k_f L \left\{ y^2 + \frac{1}{12} L^2 \theta^2 \right\} \right) = 0 \quad (A-10)$$

$$\rho L \ddot{y} + J \ddot{\theta} + k_f L y + \frac{1}{12} k_f L^3 \ddot{\theta} = 0 \quad (\text{A-11})$$

$$\{\rho \ddot{y} + k_f y\} \dot{y} = 0 \quad (\text{A-14})$$

$$\left\{ J \ddot{\theta} + \frac{1}{12} k_f L^3 \ddot{\theta} \right\} \dot{\theta} = 0 \quad (\text{A-15})$$

$$\begin{bmatrix} \rho L & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} + k_f L \begin{bmatrix} 1 & 0 \\ 0 & L^2/12 \end{bmatrix} \begin{bmatrix} y \\ \theta \end{bmatrix} = 0 \quad (\text{A-16})$$

$$\left\{ k_f L \begin{bmatrix} 1 & 0 \\ 0 & L^2/12 \end{bmatrix} - \omega^2 \begin{bmatrix} \rho L & 0 \\ 0 & J \end{bmatrix} \right\} \begin{bmatrix} y \\ \theta \end{bmatrix} = 0 \quad (\text{A-17})$$

$$\omega_1 = \sqrt{\frac{k_f}{\rho}} \quad (\text{A-18})$$

$$\omega_2 = \sqrt{\frac{k_f L^3}{12J}} \quad (\text{A-19})$$

Note that for a uniform beam,

$$J = \frac{1}{12} \rho L^3 \quad (\text{A-20})$$