DEFLECTION AND STRESS OF A BEAM UNDER STATIC LOADING
Revision A

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August 4, 2000

Introduction

The Rayleigh method uses static displacement functions to determine the natural frequencies of certain systems.

This tutorial derives static displacement functions to support vibration analysis via the Rayleigh method.

Simply-supported Beam subjected to a Concentrated Load

Consider the beam in Figure A-1.

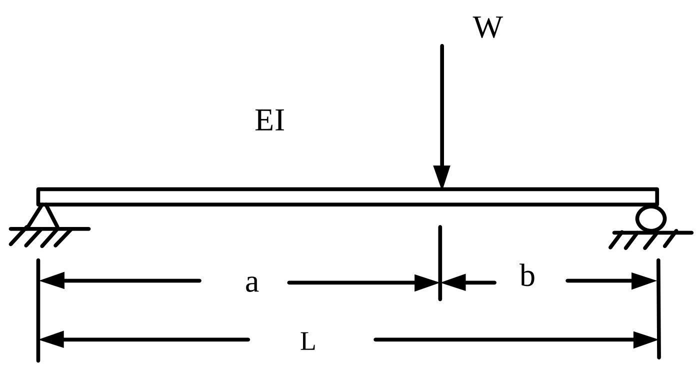


Figure A-1.

E is the modulus of elasticity.
I is the area moment of inertia.
L is the length.
W is the applied load.

Neglect the mass of the beam.

The free-body diagram of the system is given in Figure A-2.

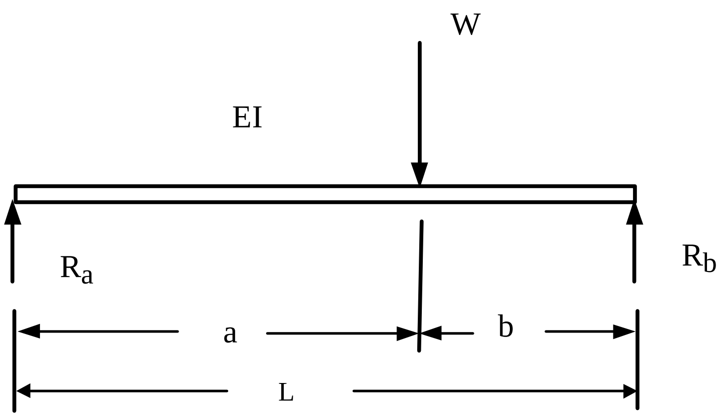


Figure A-2.

R_a and R_b are reaction forces.

Apply Newton's law for static equilibrium.

$$+\uparrow \sum \text{forces} = 0 \tag{A-1}$$

$$R_a + R_b - W = 0 \tag{A-2}$$

$$R_a = W - R_b \tag{A-3}$$

At the left boundary,

$$\curvearrowright + \sum \text{moments} = 0 \tag{A-4}$$

$$R_b L - W a = 0 \tag{A-5}$$

$$R_b L = W a \tag{A-6}$$

$$R_b = W \frac{a}{L} \tag{A-7}$$

$$R_a = W - W \frac{a}{L} \tag{A-8}$$

$$R_a = W\left[1 - \frac{a}{L}\right] \quad \text{(A-9)}$$

$$R_a = W\left[\frac{L-a}{L}\right] \quad \text{(A-10)}$$

$$R_a = W\frac{b}{L} \quad \text{(A-11)}$$

Now consider a segment of the beam, starting from the left boundary.

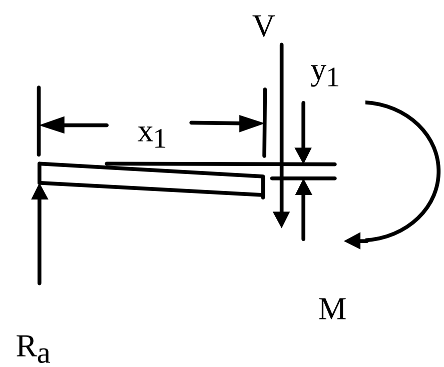


Figure A-3.

V is the shear force.
M is the bending moment.
y_1 is the deflection at position x_1.

Sum the moments at the right side of the segment, for $x_1 \leq a$.

$$\curvearrowright + \sum \text{moments} = 0 \quad \text{(A-12)}$$

$$-M - R_a x_1 = 0 \quad \text{(A-13)}$$

$$M = -R_a x_1 \quad \text{(A-14)}$$

The moment M and the deflection y are related by the equation

$$M = EIy_1'' \quad \text{(A-15)}$$

$$EIy_1'' = -R_a x_1 \tag{A-16}$$

$$y_1'' = -\frac{R_a}{EI} x_1 \tag{A-17}$$

Integrating,

$$y_1' = -\frac{R_a}{2EI}\, x_1^2 + c_1 \tag{A-18}$$

Integrating again,

$$y_1 = -\frac{R_a}{6EI}\, x_1^3 + c_1 x_1 + c_2 \tag{A-19}$$

A boundary condition at the left end is

$$y_1(0) = 0 \qquad \text{(zero displacement)} \tag{A-20}$$

Thus

$$c_2 = 0 \tag{A-21}$$

$$y_1 = -\frac{R_a}{6EI}\, x_1^3 + c_1 x_1 \tag{A-22}$$

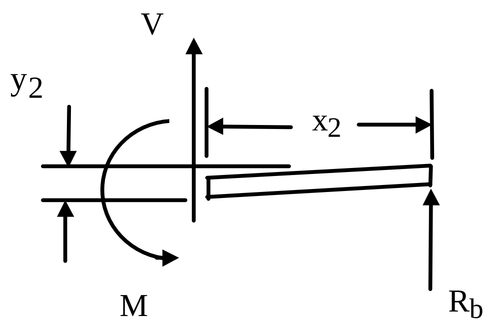
The displacement function is thus

$$y_1(x_1) = -\frac{Wb}{6EIL}\, x_1^3 + c_1 x_1 , \quad 0 \le x_1 \le a \tag{A-23}$$

The slope equation is

$$y_1'(x_1) = -\frac{Wb}{2EIL}\, x_1^2 + c_1 , \quad 0 \le x_1 \le a \tag{A-24}$$

Consider the beam segment on the right side of the applied load.



Sum the moments at the left side of the segment, for $x_2 \leq b$.

$$\circlearrowright + \sum \text{moments} = 0 \tag{A-25}$$

$$M + R_b\, x_2 = 0 \tag{A-26}$$

$$M = - R_b\, x_2 \tag{A-27}$$

The moment M and the deflection y are related by the equation

$$M = EIy_2'' \tag{A-28}$$

$$EIy_2'' = -R_b\, x_2 \tag{A-29}$$

$$y_2'' = -\frac{R_b}{EI} x_2 \tag{A-30}$$

Integrating,

$$y_2' = -\frac{R_b}{2EI}\, x_2^2 + c_3 \tag{A-31}$$

Integrating again,

$$y_2 = -\frac{R_b}{6EI}\ x_2{}^3 + c_3 x_2 + c_4 \tag{A-32}$$

A boundary condition at the right end is

$$y_2(0) = 0 \qquad \text{(zero displacement)} \tag{A-33}$$

Thus

$$c_3 = 0 \tag{A-34}$$

$$y_2 = -\frac{R_b}{6EI}\ x_2{}^3 + c_3 x_2 \tag{A-35}$$

The displacement function is thus

$$y_2(x_2) = -\frac{Wa}{6EIL}\ x_2{}^3 + c_3 x_2\ , \quad 0 \le x_2 \le b \tag{A-36}$$

The slope equation is

$$y_2{}'(x_2) = -\frac{Wa}{2EIL}\ x_2{}^2 + c_3\ , \quad 0 \le x_2 \le b \tag{A-37}$$

The boundary condition at the load point is

$$y_1(a) = y_2(b) \tag{A-38}$$

Apply the boundary condition to equations (A-23) and (A-36).

$$-\frac{Wb}{6EIL}\ a^3 + c_1 a = -\frac{Wa}{6EIL}\ b^3 + c_3 b \tag{A-39}$$

$$c_1 a - c_3 b = -\frac{Wa}{6EIL}\ b^3 + \frac{Wb}{6EIL}\ a^3 \tag{A-40}$$

$$c_1 a - c_3 b = \frac{Wab}{6EIL}\left[a^2 - b^2\right] \tag{A-41}$$

$$c_1 a = c_3 b + \frac{Wab}{6EIL}\left[a^2 - b^2\right] \quad \text{(A-42)}$$

$$c_1 = c_3 \frac{b}{a} + \frac{Wb}{6EIL}\left[a^2 - b^2\right] \quad \text{(A-43)}$$

The second boundary condition at the load point is

$$y_1'(a) = -y_2'(b) \quad \text{(A-44)}$$

Apply the boundary condition to equations (A-24) and (A-37).

$$-\frac{Wb}{2EIL}\ a^2 + c_1 = \frac{Wa}{2EIL}\ b^2 - c_3 \quad \text{(A-45)}$$

$$c_1 + c_3 = \frac{Wa}{2EIL}\ b^2 + \frac{Wb}{2EIL}\ a^2 \quad \text{(A-46)}$$

$$c_1 + c_3 = \frac{Wab}{2EIL}\ (a + b) \quad \text{(A-47)}$$

$$c_1 = -c_3 + \frac{Wab}{2EIL}\ (a + b) \quad \text{(A-48)}$$

$$c_1 = c_3 \frac{b}{a} + \frac{Wb}{6EIL}\left[a^2 - b^2\right] \quad \text{(A-49)}$$

$$-c_3 + \frac{Wab}{2EIL}\ (a + b) = c_3 \frac{b}{a} + \frac{Wb}{6EIL}\left[a^2 - b^2\right] \quad \text{(A-50)}$$

$$c_3 \frac{b}{a} + c_3 = \frac{Wab}{2EIL}\ (a + b) - \frac{Wb}{6EIL}\left[a^2 - b^2\right] \quad \text{(A-51)}$$

$$c_3 b + c_3 a = \frac{Wa^2b}{2EIL}\ (a + b) - \frac{Wab}{6EIL}\left[a^2 - b^2\right] \quad \text{(A-52)}$$

$$c_3(a+b)=\left\{\frac{Wab}{EIL}\right\}\left\{\frac{a}{2}\,(a+b)-\frac{1}{6}\left(a^2-b^2\right)\right\} \tag{A-53}$$

$$c_3(a+b)=\left\{\frac{Wab}{6EIL}\right\}\left\{3a\,(a+b)-\left(a^2-b^2\right)\right\} \tag{A-54}$$

$$c_3(a+b)=\left\{\frac{Wab}{6EIL}\right\}\left\{3a^2+3ab-a^2+b^2\right\} \tag{A-55}$$

$$c_3(a+b)=\left\{\frac{Wab}{6EIL}\right\}\left\{2a^2+3ab+b^2\right\} \tag{A-56}$$

$$c_3(a+b)=\left\{\frac{Wab}{6EIL}\right\}\{2a+b\}\{a+b\} \tag{A-57}$$

$$c_3=\left\{\frac{Wab}{6EIL}\right\}\{2a+b\} \tag{A-58}$$

Substitute equation (A-58) into (A-48).

$$c_1=-\left\{\frac{Wab}{6EIL}\right\}\{2a+b\}+\frac{Wab}{2EIL}\,(a+b) \tag{A-59}$$

$$c_1=\left\{\frac{Wab}{6EIL}\right\}\{-\{2a+b\}+3\,(a+b)\} \tag{A-60}$$

$$c_1=\left\{\frac{Wab}{6EIL}\right\}\{-2a-b+3a+3b\} \tag{A-61}$$

$$c_1=\left\{\frac{Wab}{6EIL}\right\}\{a+2b\} \tag{A-62}$$

$$y_1(x_1)=-\frac{Wb}{6EIL}\,x_1^3+\left\{\frac{Wab}{6EIL}\right\}\{a+2b\}x_1\,,\quad 0\le x_1\le a \tag{A-63}$$

$$y_1(x_1)=\left\{\frac{Wb}{6EIL}\right\}\left\{-x_1^3+a[a+2b]\,x_1\right\},\quad 0\le x_1\le a \tag{A-64}$$

$$y_1(a) = \left\{\frac{Wb}{6EIL}\right\}\left\{-a^3 + a^2[a + 2b]\right\} \quad \text{(A-65)}$$

$$y_1(a) = \left\{\frac{Wb}{6EIL}\right\}\left\{-a^3 + a^3 + 2a^2 b\right\} \quad \text{(A-66)}$$

$$y_1(a) = \left\{\frac{Wb}{6EIL}\right\}\left\{2a^2 b\right\} \quad \text{(A-67)}$$

$$y_1(a) = \left\{\frac{Wa^2 b^2}{3EIL}\right\} \quad \text{(A-68)}$$

$$y_2(x_2) = -\frac{Wa}{6EIL}\ x_2{}^3 + \left\{\frac{Wab}{6EIL}\right\}\{2a + b\}x_2\ , \quad 0 \le x_2 \le b \quad \text{(A-69)}$$

$$y_2(x_2) = \left\{\frac{Wa}{6EIL}\right\}\left\{-\ x_2{}^3 + b\{2a + b\}x_2\right\}, \quad 0 \le x_2 \le b \quad \text{(A-70)}$$

$$y_2(b) = \left\{\frac{Wa}{6EIL}\right\}\left\{-\ b^3 + b^2[2a + b]\ \right\} \quad \text{(A-71)}$$

$$y_2(b) = \left\{\frac{Wa}{6EIL}\right\}\left\{-\ b^3 + 2ab^2 + b^3\ \right\} \quad \text{(A-72)}$$

$$y_2(b) = \left\{\frac{Wa}{6EIL}\right\}\left\{2ab^2\ \right\} \quad \text{(A-73)}$$

$$y_2(b) = \left\{\frac{Wa^2 b^2}{3EIL}\right\} \quad \text{(A-74)}$$

Note that the displacement in equation (A-68) is the same as in (A-74), as expected from the intermediate boundary condition. The stiffness k at the load point is thus

$$k = \left\{\frac{3EIL}{a^2 b^2}\right\} \quad \text{(A-75)}$$

Bending Moment and Bending Stress

Consider the bending moment for the left segment. Recall that the moment M and the deflection y are related by the equation

$$M = EI y_1'' \quad \text{(A-76)}$$

$$EI y_1'' = -R_a x_1 \quad \text{(A-77)}$$

$$y_1'' = -\frac{R_a}{EI} x_1 \quad \text{(A-78)}$$

$$R_a = W \frac{b}{L} \quad \text{(A-79)}$$

$$y_1'' = -W \frac{b}{EIL} x_1 \quad \text{(A-80)}$$

$$M = EI\left\{-W \frac{b}{EIL} x_1\right\} \quad \text{(A-81)}$$

$$M_1 = -W \frac{b}{L} x_1 \quad \text{(A-82)}$$

The maximum bending stress, or fiber stress, σ occurs at the point, or points, furthest from the neutral axis. Let c be the distance to the furthest point from the neutral axis.

$$\sigma = \frac{M c}{I} \quad \text{(A-83)}$$

$$\sigma(x_1) = \left[-W \frac{b}{L} x_1\right]\left[\frac{c}{I}\right] \quad \text{(A-84)}$$

The maximum bending stress at the load point is

$$\sigma(a) = \left[-W \frac{a b}{L}\right]\left[\frac{c}{I}\right] \quad \text{(A-85)}$$

Repeat the bending stress analysis for the right segment.

The moment M and the deflection y are related by the equation

$$M = EIy_2'' \tag{A-86}$$

$$EIy_2'' = -R_b x_2 \tag{A-87}$$

$$y_2'' = -\frac{R_b}{EI} x_2 \tag{A-88}$$

$$R_b = W\frac{a}{L} \tag{A-89}$$

$$y_2'' = -\left[W\frac{a}{L}\right]\left[\frac{1}{EI} x_2\right] \tag{A-90}$$

$$M = -EI\left[W\frac{a}{L}\right]\left[\frac{1}{EI} x_2\right] \tag{A-91}$$

$$M_2 = -W\frac{a}{L} x_2 \tag{A-92}$$

$$\sigma = \frac{M\,c}{I} \tag{A-93}$$

$$\sigma(x_2) = \left[-W\frac{a}{L} x_2\right]\left[\frac{c}{I}\right] \tag{A-94}$$

Shear Force and Shear Stress

The shear force V is related to the bending moment M by

$$V = \frac{d}{dx} M \tag{A-95}$$

The shear force for the left segment is

$$V_1 = \frac{d}{dx}\left\{- W \frac{b}{L} x_1\right\} \tag{A-96}$$

$$V_1 = -W \frac{b}{L} \tag{A-97}$$

The shear force for the right segment is

$$V_2 = \frac{d}{dx}\left\{- W \frac{a}{L} x_2\right\} \tag{A-98}$$

$$V_2 = -W \frac{a}{L} \tag{A-99}$$

The maximum shear stress in the beam occurs at the section of greatest vertical shear. The maximum shear stress at any section occurs at the neutral axis, assuming a uniform cross-section.

The maximum shear stress τ is given by

$$\tau = \alpha \frac{V}{A} \tag{A-100}$$

The term V/A is the average shear stress on the section. The coefficient α is a factor that depends on the cross-section form.

The shear factors for two cross-sections are given in Table A-1.

Table A-1. Shear Factors	
Cross-Section	Shear Factor α
Rectangular	2/3
Solid Circular	3/4