TRANSVERSE VIBRATION OF A FIXED-FREE TIMOSHENKO BEAM

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Variables

$y(x, t)$	=	Transverse displacement
E	=	Elastic modulus
G	=	Shear modulus
I	=	Area moment of inertia
A	=	Cross-section area
r	=	Radius of gyration
k	=	Shear factor
ρ	=	Mass/volume

Note the shear modulus can be expressed as

$$G = \frac{E}{2(1+\nu)} \tag{1}$$

where ν is the Poisson's ratio.

Equation of Motion

Consider the bending vibration of a beam including both shear and rotary inertia effects. Assume that the beam is uniform.

The following coupled partial differential equations are taken from Reference 1.

$$\rho A \frac{\partial^2 v}{\partial t^2} - kAG\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial \theta}{\partial x}\right) = 0 \tag{2}$$

$$EI\, \frac{\partial^2 \theta}{\partial x^2} + kAG\left(\frac{\partial v}{\partial x} - \theta\right) - I\rho \frac{\partial^2 \theta}{\partial t^2} = 0 \tag{3}$$

Assume separation of variables.

$$v(x,t) = V(x)\exp(j\omega t) \tag{4}$$

$$\frac{\partial}{\partial t} v = j\omega V(x)\exp(j\omega t) \tag{5}$$

$$\frac{\partial^2}{\partial t^2} v = -\omega^2\, V(x)\exp(j\omega t) \tag{6}$$

$$\theta(x,t) = \Theta(x)\exp(j\omega t) \tag{7}$$

$$\frac{\partial}{\partial t} \theta = j\omega \Theta(x)\exp(j\omega t) \tag{8}$$

$$\frac{\partial^2}{\partial t^2} \theta = -\omega^2\, \Theta(x)\exp(j\omega t) \tag{9}$$

By substitution,

$$-\omega^2 \rho A\ V - kAG\left(\frac{d^2}{dx^2} V - \frac{d}{dx}\Theta\right) = 0 \tag{10}$$

$$-\omega^2 \rho A\ V - kAG \frac{d^2}{dx^2} V + kAG \frac{d}{dx}\Theta = 0 \tag{11}$$

$$-\omega^2\rho\, V - kG\frac{d^2}{dx^2}V + kG\frac{d}{dx}\Theta = 0 \tag{12}$$

$$EI\frac{d^2}{dx^2}\Theta + kAG\left(\frac{d}{dx}V - \Theta\right) + \omega^2 I\rho\Theta = 0 \tag{13}$$

$$EI\frac{d^2}{dx^2}\Theta + kAG\frac{d}{dx}V + \left[\omega^2 I\rho - kAG\right]\Theta = 0 \tag{14}$$

Define a non-dimensional spatial variable.

$$\xi = \frac{x}{L} \tag{15}$$

$$x = \xi L \tag{16}$$

$$dx = L d\xi \tag{17}$$

$$\frac{d}{dx} = \frac{d\xi}{dx}\frac{d}{d\xi} \tag{18}$$

$$\frac{d}{dx} = \frac{1}{L}\frac{d}{d\xi} \tag{19}$$

The pair of equations of motion becomes

$$-\omega^2\rho\, V - \frac{kG}{L^2}\frac{d^2}{d\xi^2}V + \frac{kG}{L}\frac{d}{d\xi}\Theta = 0 \tag{20}$$

$$\frac{EI}{L^2}\frac{d^2}{d\xi^2}\Theta + \frac{kAG}{L}\frac{d}{d\xi}V + \left[\omega^2 I\rho - kAG\right]\Theta = 0 \tag{21}$$

Take the derivative of equation (22).

$$\frac{EI}{L^2}\frac{d^3}{d\xi^3}\Theta + \frac{kAG}{L}\frac{d^2}{d\xi^2}V + \left[\omega^2 I\rho - kAG\right]\frac{d}{d\xi}\Theta = 0 \tag{22}$$

$$\frac{d}{d\xi}\Theta = \frac{\omega^2\rho L}{kG}V + \frac{1}{L}\frac{d^2}{d\xi^2}V \tag{23}$$

$$\frac{d^2}{d\xi^2}\Theta = \frac{\omega^2\rho L}{kG}\frac{d}{d\xi}V + \frac{1}{L}\frac{d^3}{d\xi^3}V \tag{24}$$

$$\frac{d^3}{d\xi^3}\Theta = \frac{\omega^2\rho L}{kG}\frac{d^2}{d\xi^2}V + \frac{1}{L}\frac{d^4}{d\xi^4}V \tag{25}$$

$$\frac{EI}{L^2}\left[\frac{\omega^2\rho L}{kG}\frac{d^2}{d\xi^2}V + \frac{1}{L}\frac{d^4}{d\xi^4}V\right] + \frac{kAG}{L}\frac{d^2}{d\xi^2}V + \left[\omega^2 I\rho - kAG\right]\left[\frac{\omega^2\rho L}{kG}V + \frac{1}{L}\frac{d^2}{d\xi^2}V\right] = 0 \tag{26}$$

$$\frac{EI}{L^3}\frac{d^4}{d\xi^4}V + \frac{\omega^2\rho EI}{kGL}\frac{d^2}{d\xi^2}V + \frac{kAG}{L}\frac{d^2}{d\xi^2}V + \left[\omega^2 I\rho - kAG\right]\frac{\omega^2\rho L}{kG}V + \left[\omega^2 I\rho - kAG\right]\frac{1}{L}\frac{d^2}{d\xi^2}V = 0 \tag{27}$$

$$\frac{EI}{L^3}\frac{d^4}{d\xi^4}V + \left[\frac{\omega^2\rho EI}{kGL} + \frac{\omega^2 I\rho}{L}\right]\frac{d^2}{d\xi^2}V + \left[\omega^2 I\rho - kAG\right]\frac{\omega^2\rho L}{kG}V = 0 \tag{28}$$

Multiply through by L^3/EI.

$$\frac{d^4}{d\xi^4}V+\left[\frac{L^3}{EI}\right]\left[\frac{\omega^2\rho EI}{kGL}+\frac{\omega^2 I\rho}{L}\right]\frac{d^2}{d\xi^2}V+\left[\frac{L^3}{EI}\right]\left[\omega^2 I\rho-kAG\right]\frac{\omega^2\rho L}{kG}V=0 \tag{29}$$

$$\frac{d^4}{d\xi^4}V+\left[\frac{L^3}{EI}\right]\left[\frac{\omega^2\rho EI}{kGL}+\frac{\omega^2 I\rho}{L}\right]\frac{d^2}{d\xi^2}V-\left[\frac{\omega^2\rho L^4}{kGEI}\right]\left[kAG-\omega^2 I\rho\right]V=0 \tag{30}$$

$$\frac{d^4}{d\xi^4}V+\left[\frac{\rho A\omega^2 L^4}{EI}\right]\left[\frac{EI}{kAGL^2}+\frac{I}{AL^2}\right]\frac{d^2}{d\xi^2}V-\left[\frac{\omega^2\rho L^4}{kGEI}\right]\left[kAG-\omega^2 I\rho\right]V=0 \tag{31}$$

$$\frac{d^4}{d\xi^4}V+\left[\frac{\rho A\omega^2 L^4}{EI}\right]\left[\frac{EI}{kAGL^2}+\frac{I}{AL^2}\right]\frac{d^2}{d\xi^2}V-\left[\frac{\rho A\omega^2 L^4}{EI}\right]\left[1-\frac{\omega^2 I\rho}{kAG}\right]V=0 \tag{32}$$

Let

$$b^2=\frac{\rho A\,\omega^2 L^4}{EI} \tag{33}$$

$$s^2=\frac{EI}{kAGL^2} \tag{34}$$

$$r^2=\frac{I}{AL^2} \tag{35}$$

Note that

$$b^2 s^2 r^2=\frac{\omega^2\, I\rho}{kAG} \tag{36}$$

$$\frac{d^4}{d\xi^4}V\quad+b^2\left\{s^2+r^2\right\}\frac{d^2}{d\xi^2}V\;-\;\left\{b^2\left[1-b^2 s^2 r^2\right]\right\}V=0 \tag{37}$$

Recall

$$-\omega^2\rho\, V - \frac{kG}{L^2}\frac{d^2}{d\xi^2}V + \frac{kG}{L}\frac{d}{d\xi}\Theta = 0 \tag{38}$$

Take the derivative.

$$-\omega^2\rho\,\frac{d}{d\xi}V - \frac{kG}{L^2}\frac{d^3}{d\xi^3}V + \frac{kG}{L}\frac{d^2}{d\xi^2}\Theta = 0 \tag{39}$$

Recall

$$\frac{EI}{L^2}\frac{d^2}{d\xi^2}\Theta + \frac{kAG}{L}\frac{d}{d\xi}V + \left[\omega^2 I\rho - kAG\right]\Theta = 0 \tag{40}$$

$$\frac{d}{d\xi}V = -\frac{EI}{kAG\,L}\frac{d^2}{d\xi^2}\Theta + \frac{L}{KAG}\left[kAG - \omega^2 I\rho\right]\Theta \tag{41}$$

$$\frac{d^2}{d\xi^2}V = -\frac{EI}{kAG\,L}\frac{d^3}{d\xi^3}\Theta + \frac{L}{KAG}\left[kAG - \omega^2 I\rho\right]\frac{d}{d\xi}\Theta \tag{42}$$

$$\frac{d^3}{d\xi^3}V = -\frac{EI}{kAG\,L}\frac{d^4}{d\xi^4}\Theta + \frac{L}{KAG}\left[kAG - \omega^2 I\rho\right]\frac{d^2}{d\xi^2}\Theta \tag{43}$$

$$-\omega^2\rho\left\{-\frac{EI}{kAG\,L}\frac{d^2}{d\xi^2}\Theta + \frac{L}{kAG}\left[kAG - \omega^2 I\rho\right]\Theta\right\}$$
$$-\frac{kG}{L^2}\left\{-\frac{EI}{kAG\,L}\frac{d^4}{d\xi^4}\Theta + \frac{L}{kAG}\left[kAG - \omega^2 I\rho\right]\frac{d^2}{d\xi^2}\Theta\right\} + \frac{kG}{L}\frac{d^2}{d\xi^2}\Theta = 0$$

(44)

$$\left\{\frac{EI\omega^2\rho}{kAGL}\frac{d^2}{d\xi^2}\Theta-\frac{L\omega^2\rho}{kAG}\left[kAG-\omega^2 I\rho\right]\Theta\right\}$$

$$+\left\{\frac{EI}{AL^3}\frac{d^4}{d\xi^4}\Theta-\frac{1}{AL}\left[kAG-\omega^2 I\rho\right]\frac{d^2}{d\xi^2}\Theta\right\}+\frac{kG}{L}\frac{d^2}{d\xi^2}\Theta=0$$

(45)

$$\frac{EI}{AL^3}\frac{d^4}{d\xi^4}\Theta+\left\{\frac{EI\omega^2\rho}{kAGL}-\frac{1}{AL}\left[kAG-\omega^2 I\rho\right]+\frac{kG}{L}\right\}\frac{d^2}{d\xi^2}\Theta-\frac{L\omega^2\rho}{kAG}\left[kAG-\omega^2 I\rho\right]\Theta=0$$

(46)

$$\frac{EI}{AL^3}\frac{d^4}{d\xi^4}\Theta+\left\{\frac{EI\omega^2\rho}{kAGL}+\frac{\omega^2 I\rho}{AL}\right\}\frac{d^2}{d\xi^2}\Theta-\frac{L\omega^2\rho}{kAG}\left[kAG-\omega^2 I\rho\right]\Theta=0 \quad (47)$$

$$\frac{d^4}{d\xi^4}\Theta+\frac{AL^3}{EI}\left\{\frac{EI\omega^2\rho}{kAGL}+\frac{\omega^2 I\rho}{AL}\right\}\frac{d^2}{d\xi^2}\Theta-\frac{L^4\omega^2\rho A}{kAG\,EI}\left[kAG-\omega^2 I\rho\right]\Theta=0 \quad (48)$$

$$\frac{d^4}{d\xi^4}\Theta+\frac{\omega^2\rho A L^4}{EI}\left\{\frac{EI}{kAGL^2}+\frac{I}{AL^2}\right\}\frac{d^2}{d\xi^2}\Theta-\frac{L^4\omega^2\rho A}{EI}\left[1-\frac{\omega^2 I\rho}{kAG}\right]\Theta=0 \quad (49)$$

By substitution,

$$\frac{d^4}{d\xi^4}\Theta+b^2\left[r^2+s^2\right]\frac{d^2}{d\xi^2}\Theta-b^2\left[1-b^2s^2r^2\right]\Theta=0 \quad (50)$$

Assume a displacement function of

$$V(\xi) = c_1 \cosh(b\alpha\xi) + c_2 \sinh(b\alpha\xi) + c_3 \cos(b\beta\xi) + c_4 \sin(b\beta\xi) \tag{51}$$

$$\frac{d}{d\xi} V(\xi) = c_1 b\alpha \sinh(b\alpha\xi) + c_2 b\alpha \cosh(b\alpha\xi) - c_3 b\beta \sin(b\beta\xi) + c_4 b\beta \cos(b\beta\xi) \tag{52}$$

$$\frac{d^2}{d\xi^2} V(\xi) = c_1 (b\alpha)^2 \cosh(b\alpha\xi) + c_2 (b\alpha)^2 \sinh(b\alpha\xi) - c_3 (b\beta)^2 \cos(b\beta\xi) - c_4 (b\beta)^2 \sin(b\beta\xi) \tag{53}$$

$$\frac{d^3}{d\psi^3} V(\xi) = c_1 (b\alpha)^3 \sinh(b\alpha\xi) + c_2 (b\alpha)^3 \cosh(b\alpha\xi) + c_3 (b\beta)^3 \sin(b\beta\xi) - c_4 (b\beta)^3 \cos(b\beta\xi) \tag{54}$$

$$\frac{d^4}{d\xi^4} V(\xi) = c_1 (b\alpha)^4 \cosh(b\alpha\xi) + c_2 (b\alpha)^4 \sinh(b\alpha\xi) + c_3 (b\beta)^4 \cos(b\beta\xi) + c_4 (b\beta)^4 \sin(b\beta\xi) \tag{55}$$

Also assume an angular displacement function of

$$\Theta(\xi) = \hat{c}_1 \cosh(b\alpha\xi) + \hat{c}_2 \sinh(b\alpha\xi) + \hat{c}_3 \cos(b\beta\xi) + \hat{c}_4 \sin(b\beta\xi) \tag{56}$$

Solve for α.

$$\begin{aligned} &\left\{c_1 (b\alpha)^4 \cosh(b\alpha\xi) + c_2 (b\alpha)^4 \sinh(b\alpha\xi) + c_3 (b\beta)^4 \cos(b\beta\xi) + c_4 (b\beta)^4 \sin(b\beta\xi)\right\} \\ &+ b^2 \left\{s^2 + r^2\right\}\left\{c_1 (b\alpha)^2 \cosh(b\alpha\xi) + c_2 (b\alpha)^2 \sinh(b\alpha\xi) - c_3 (b\beta)^2 \cos(b\beta\xi) - c_4 (b\beta)^2 \sin(b\beta\xi)\right\} \\ &- \left\{b^2\left[1 - b^2 s^2 r^2\right]\right\}\left\{c_1 \cosh(b\alpha\xi) + c_2 \sinh(b\alpha\xi) + c_3 \cos(b\beta\xi) + c_4 \sin(b\beta\xi)\right\} = 0 \end{aligned} \tag{57}$$

$$b^4 \alpha^4 + b^4 \left\{s^2 + r^2\right\}\alpha^2 - \left\{b^2\left[1 - b^2 s^2 r^2\right]\right\} = 0 \tag{58}$$

$$\alpha^2 = \frac{1}{2b^4}\left\{-b^4\left\{s^2+r^2\right\}\pm\sqrt{b^8\left\{s^2+r^2\right\}^2+4b^6\left\{1-b^2s^2r^2\right\}}\right\} \tag{59}$$

$$\alpha^2 = \frac{1}{2}\left\{-\left\{s^2+r^2\right\}\pm\sqrt{\left\{s^2+r^2\right\}^2+\frac{4}{b^2}\left\{1-b^2s^2r^2\right\}}\right\} \tag{60}$$

$$\alpha^2 = \frac{1}{2}\left\{-\left\{s^2+r^2\right\}\pm\sqrt{s^4+2s^2r^2+r^4+\frac{4}{b^2}-4s^2r^2}\right\} \tag{61}$$

$$\alpha^2 = \frac{1}{2}\left\{-\left\{s^2+r^2\right\}\pm\sqrt{s^4-2s^2r^2+r^4+\frac{4}{b^2}}\right\} \tag{62}$$

$$\alpha^2 = \frac{1}{2}\left\{-\left(r^2+s^2\right)\pm\sqrt{\left(r^2-s^2\right)^2+\frac{4}{b^2}}\right\} \tag{63}$$

$$\alpha = \pm\frac{1}{\sqrt{2}}\sqrt{-\left(r^2+s^2\right)\pm\sqrt{\left(r^2-s^2\right)^2+\frac{4}{b^2}}} \tag{64}$$

Solve for β.

$$\begin{aligned}&\left\{c_1(b\alpha)^4\cosh(b\alpha\xi)+c_2(b\alpha)^4\sinh(b\alpha\xi)+c_3(b\beta)^4\cos(b\beta\xi)+c_4(b\beta)^4\sin(b\beta\xi)\right\}\\&+b^2\left\{s^2+r^2\right\}\left\{c_1(b\alpha)^2\cosh(b\alpha\xi)+c_2(b\alpha)^2\sinh(b\alpha\xi)-c_3(b\beta)^2\cos(b\beta\xi)-c_4(b\beta)^2\sin(b\beta\xi)\right\}\\&-\left\{b^2\left[1-b^2s^2r^2\right]\right\}\left\{c_1\cosh(b\alpha\xi)+c_2\sinh(b\alpha\xi)+c_3\cos(b\beta\xi)+c_4\sin(b\beta\xi)\right\}=0\end{aligned} \tag{65}$$

$$b^4\beta^4 \quad - b^4\left\{s^2 + r^2\right\}\beta^2 - \left\{b^2\left[1 - b^2 s^2 r^2\right]\right\} = 0 \tag{66}$$

$$\beta^2 = \frac{1}{2}\left\{\left\{s^2 + r^2\right\} \pm \sqrt{\left\{s^2 + r^2\right\}^2 + \frac{4}{b^2}\left\{1 - b^2 s^2 r^2\right\}}\right\} \tag{67}$$

$$\beta^2 = \frac{1}{2}\left\{\left\{s^2 + r^2\right\} \pm \sqrt{s^4 + 2s^2 r^2 + r^4 + \frac{4}{b^2} - 4s^2 r^2}\right\} \tag{68}$$

$$\beta^2 = \frac{1}{2}\left\{\left\{s^2 + r^2\right\} \pm \sqrt{s^4 - 2s^2 r^2 + r^4 + \frac{4}{b^2}}\right\} \tag{69}$$

$$\beta^2 = \frac{1}{2}\left\{\left(r^2 + s^2\right) \pm \sqrt{\left(r^2 - s^2\right)^2 + \frac{4}{b^2}}\right\} \tag{70}$$

$$\beta = \pm\frac{1}{\sqrt{2}}\sqrt{\left(r^2 + s^2\right) \pm \sqrt{\left(r^2 - s^2\right)^2 + \frac{4}{b^2}}} \tag{71}$$

Require real, positive values. In summary,

$$\begin{Bmatrix}\alpha \\ \beta\end{Bmatrix} = \frac{1}{\sqrt{2}}\sqrt{\mp\left(r^2 + s^2\right) + \sqrt{\left(r^2 - s^2\right)^2 + \frac{4}{b^2}}} \tag{72}$$

Further follow the branch such that

$$\omega^2 < kGA/(\rho I) \tag{73}$$

$$\sqrt{\left(r^2 - s^2\right)^2 + \frac{4}{b^2}} > \left(r^2 + s^2\right) \tag{74}$$

Equation (74) expresses the dispersion relationship.

Recall the displacement functions.

$$V(\xi) = c_1 \cosh(b\alpha\xi) + c_2 \sinh(b\alpha\xi) + c_3 \cos(b\beta\xi) + c_4 \sin(b\beta\xi) \tag{75}$$

$$\frac{d}{d\xi} V(\xi) = c_1(b\alpha)\sinh(b\alpha\xi) + c_2(b\alpha)\cosh(b\alpha\xi) - c_3(b\beta)\sin(b\beta\xi) + c_4(b\beta)\cos(b\beta\xi) \tag{76}$$

$$\frac{d^2}{d\xi^2} V(\xi) = c_1(b\alpha)^2 \cosh(b\alpha\xi) + c_2(b\alpha)^2 \sinh(b\alpha\xi) - c_3(b\beta)^2 \cos(b\beta\xi) - c_4(b\beta)^2 \sin(b\beta\xi) \tag{77}$$

$$\Theta(\xi) = \hat{c}_1 \sinh(b\alpha\xi) + \hat{c}_2 \cosh(b\alpha\xi) + \hat{c}_3 \sin(b\beta\xi) + \hat{c}_4 \cos(b\beta\xi) \tag{78}$$

$$\frac{d}{d\xi} \Theta(\xi) = \hat{c}_1 b\alpha \cosh(b\alpha\xi) + \hat{c}_2 b\alpha \sinh(b\alpha\xi) + \hat{c}_3 b\beta \cos(b\beta\xi) - \hat{c}_4 b\beta \sin(b\beta\xi) \tag{79}$$

$$\frac{d^2}{d\xi^2} \Theta(\xi) = \hat{c}_1(b\alpha)^2 \sinh(b\alpha\xi) + \hat{c}_2(b\alpha)^2 \cosh(b\alpha\xi) - \hat{c}_3(b\beta)^2 \sin(b\beta\xi) - \hat{c}_4(b\beta)^2 \cos(b\beta\xi) \tag{80}$$

Recall

$$-\omega^2\rho\, V - \frac{kG}{L^2}\frac{d^2}{d\xi^2}V + \frac{kG}{L}\frac{d}{d\xi}\Theta = 0 \tag{81}$$

$$\frac{d^2}{d\xi^2}V + \omega^2\rho\,\frac{L^2}{kG}V - L\frac{d}{d\xi}\Theta = 0 \tag{82}$$

$$\frac{d^2}{d\xi^2}V + b^2 s^2\, V - L\frac{d}{d\xi}\Theta = 0 \tag{83}$$

Furthermore,

$$\frac{EI}{L^2}\frac{d^2}{d\xi^2}\Theta + \frac{kAG}{L}\frac{d}{d\xi}V + \left[\omega^2 I\rho - kAG\right]\Theta = 0 \tag{84}$$

$$\frac{EI}{kAGL^2}\frac{d^2}{d\xi^2}\Theta + \left[1 - \frac{\omega^2 I\rho}{kAG}\right]\Theta + \frac{1}{L}\frac{d}{d\xi}V = 0 \tag{85}$$

$$s^2\frac{d^2}{d\xi^2}\Theta + \left[1 - b^2 r^2 s^2\right]\Theta + \frac{1}{L}\frac{d}{d\xi}V = 0 \tag{86}$$

$$\frac{d^2}{d\xi^2}V + b^2 s^2\, V - L\frac{d}{d\xi}\Theta = 0 \tag{87}$$

$$c_1(b\alpha)^2\cosh(b\alpha\xi)+c_2(b\alpha)^2\sinh(b\alpha\xi)-c_3(b\beta)^2\cos(b\beta\xi)-c_4(b\beta)^2\sin(b\beta\xi)$$
$$+b^2s^2\ \{c_1\cosh(b\alpha\xi)+c_2\sinh(b\alpha\xi)+c_3\cos(b\beta\xi)+c_4\sin(b\beta\xi)\}$$
$$-L\{\hat{c}_1 b\alpha\cosh(b\alpha\xi)+\hat{c}_2 b\alpha\sinh(b\alpha\xi)+\hat{c}_3 b\beta\cos(b\beta\xi)-\hat{c}_4 b\beta\sin(b\beta\xi)\}=0 \tag{88}$$

$$\cosh(b\alpha\xi)\{c_1(b\alpha)^2+b^2s^2c_1-L\hat{c}_1 b\alpha\}$$
$$+\sinh(b\alpha\xi)\{c_2(b\alpha)^2+b^2s^2c_2-L\hat{c}_2 b\alpha\}$$
$$+\cos(b\beta\xi)\{-c_3(b\beta)^2+b^2s^2c_3-L\hat{c}_3 b\beta\}$$
$$+\sin(b\beta\xi)\{-c_4(b\beta)^2+b^2s^2c_4+L\hat{c}_4 b\beta\}=0 \tag{89}$$

$$c_1(b\alpha)^2+b^2s^2c_1-L\hat{c}_1 b\alpha=0 \tag{90}$$

$$c_2(b\alpha)^2+b^2s^2c_2-L\hat{c}_2 b\alpha=0 \tag{91}$$

$$-c_3(b\beta)^2+b^2s^2c_3-L\hat{c}_3 b\beta=0 \tag{92}$$

$$-c_4(b\beta)^2+b^2s^2c_4+L\hat{c}_4 b\beta=0 \tag{93}$$

$$c_1 b\alpha^2+bs^2c_1-L\hat{c}_1\alpha=0 \tag{94}$$

$$c_2 b\alpha^2+bs^2c_2-L\hat{c}_2\alpha=0 \tag{95}$$

$$-c_3 b\beta^2+bs^2c_3-L\hat{c}_3\beta=0 \tag{96}$$

$$-c_4 b\beta^2+bs^2c_4+L\hat{c}_4\beta=0 \tag{97}$$

$$b\left[\alpha^2 + s^2\right]c_1 - L\hat{c}_1\alpha = 0 \qquad (98)$$

$$b\left[\alpha^2 + s^2\right]c_2 - L\hat{c}_2\alpha = 0 \qquad (99)$$

$$b\left[-\beta^2 + s^2\right]c_3 - L\hat{c}_3\beta = 0 \qquad (100)$$

$$b\left[-\beta^2 + s^2\right]c_4 + L\hat{c}_4\beta = 0 \qquad (101)$$

The final set of coefficients is

$$\hat{c}_1 = \frac{b}{L\alpha}\left[\alpha^2 + s^2\right]c_1 \qquad (102)$$

$$\hat{c}_2 = \frac{b}{L\alpha}\left[\alpha^2 + s^2\right]c_2 \qquad (103)$$

$$\hat{c}_3 = \frac{b}{L\beta}\left[-\beta^2 + s^2\right]c_3 \qquad (104)$$

$$\hat{c}_4 = \frac{b}{L\beta}\left[\beta^2 - s^2\right]c_4 \qquad (105)$$

Boundary Conditions

The boundary conditions for a fixed-free beam are

$$v(0,t) = 0 \qquad (106)$$

$$\theta(0,t) = 0 \qquad (107)$$

$$kGA\left[\theta(L,t) - \frac{\partial}{\partial x}v(L,t)\right] = 0 \qquad (108)$$

$$EI\frac{\partial}{\partial x}\theta(L,t) = 0 \quad (109)$$

Recall,

$$v(x,t) = V(x)\exp(j\omega t) \quad (110)$$

$$\theta(x,t) = \Theta(x)\exp(j\omega t) \quad (111)$$

By substitution,

$$V(0) = 0 \quad (112)$$

$$\Theta(0) = 0 \quad (113)$$

$$\Theta(L) - \frac{d}{dx}V(L) = 0 \quad (114)$$

$$\frac{d}{dx}\Theta(L) = 0 \quad (115)$$

Recall

$$\xi = \frac{x}{L} \quad (116)$$

The boundary conditions can be rewritten as

At $\xi = 0$, $(x = 0)$,

$$V(0) = 0 \quad (117)$$

$$\Theta(0) = 0 \quad (118)$$

At $\xi = 1$, $(x = L)$,

$$\Theta(1) - \frac{1}{L}\frac{d}{d\xi}V(1) = 0 \tag{119}$$

$$\frac{d}{d\xi}\Theta(1) = 0 \tag{120}$$

First boundary condition,

$$c_1 + c_3 = 0 \tag{121}$$

Second boundary condition,

$$\hat{c}_2 + \hat{c}_4 = 0 \tag{122}$$

$$\frac{b}{L\alpha}\left[\alpha^2 + s^2\right]c_2 + \frac{b}{L\beta}\left[\beta^2 - s^2\right]c_4 = 0 \tag{123}$$

$$\frac{1}{\alpha}\left[\alpha^2 + s^2\right]c_2 + \frac{1}{\beta}\left[\beta^2 - s^2\right]c_4 = 0 \tag{124}$$

Third boundary condition,

$$\Theta(1) - \frac{1}{L}\frac{d}{d\xi}V(1) = 0 \tag{125}$$

$$\begin{aligned}&\hat{c}_1\sinh(b\alpha) + \hat{c}_2\cosh(b\alpha) + \hat{c}_3\sin(b\beta) + \hat{c}_4\cos(b\beta)\\&-\frac{1}{L}\{c_1(b\alpha)\sinh(b\alpha) + c_2(b\alpha)\cosh(b\alpha) - c_3(b\beta)\sin(b\beta) + c_4(b\beta)\cos(b\beta)\} = 0\end{aligned} \tag{126}$$

$$\hat{c}_1 \sinh(b\alpha) + \hat{c}_2 \cosh(b\alpha) + \hat{c}_3 \sin(b\beta) + \hat{c}_4 \cos(b\beta)$$
$$+ \frac{1}{L}\{-c_1(b\alpha)\sinh(b\alpha) - c_2(b\alpha)\cosh(b\alpha) + c_3(b\beta)\sin(b\beta) - c_4(b\beta)\cos(b\beta)\} = 0 \tag{127}$$

$$\left[\hat{c}_1 - \frac{1}{L}(b\alpha)c_1\right]\sinh(b\alpha) + \left[\hat{c}_2 - \frac{1}{L}(b\alpha)c_2\right]\cosh(b\alpha)$$
$$+ \left[\hat{c}_3 + \frac{1}{L}(b\beta)c_3\right]\sin(b\beta) + \left[\hat{c}_4 - \frac{1}{L}(b\beta)c_4\right]\cos(b\beta) = 0 \tag{128}$$

$$[L\hat{c}_1 - (b\alpha)c_1]\sinh(b\alpha) + [L\hat{c}_2 - (b\alpha)c_2]\cosh(b\alpha)$$
$$+ [L\hat{c}_3 + (b\beta)c_3]\sin(b\beta) + [L\hat{c}_4 - (b\beta)c_4]\cos(b\beta) = 0 \tag{129}$$

$$\left[\frac{b}{\alpha}\left[\alpha^2 + s^2\right]c_1 - (b\alpha)c_1\right]\sinh(b\alpha)$$
$$+ \left[\frac{b}{\alpha}\left[\alpha^2 + s^2\right]c_2 - (b\alpha)c_2\right]\cosh(b\alpha)$$
$$+ \left[\frac{b}{\beta}\left[-\beta^2 + s^2\right]c_3 + (b\beta)c_3\right]\sin(b\beta)$$
$$+ \left[\frac{b}{\beta}\left[\beta^2 - s^2\right]c_4 - (b\beta)c_4\right]\cos(b\beta) = 0 \tag{130}$$

$$\left[\frac{b}{\alpha}\left[\alpha^2 + s^2\right] - \frac{1}{\alpha}\left(b\alpha^2\right)\right]c_1 \sinh(b\alpha)$$
$$+ \left[\frac{b}{\alpha}\left[\alpha^2 + s^2\right] - \frac{1}{\alpha}\left(b\alpha^2\right)\right]c_2 \cosh(b\alpha)$$
$$+ \left[\frac{b}{\beta}\left[-\beta^2 + s^2\right] + \frac{1}{\beta}\left(b\beta^2\right)\right]c_3 \sin(b\beta)$$
$$+ \left[\frac{b}{\beta}\left[\beta^2 - s^2\right] - \frac{1}{\beta}\left(b\beta^2\right)\right]c_4 \cos(b\beta) = 0 \tag{131}$$

$$\frac{b}{\alpha}s^2 c_1 \sinh(b\alpha) + \frac{b}{\alpha}s^2 c_2 \cosh(b\alpha) + \frac{b}{\beta}s^2 c_3 \sin(b\beta) - \frac{b}{\beta}s^2 c_4 \cos(b\beta) = 0 \tag{132}$$

$$\frac{b}{\alpha}c_1 \sinh(b\alpha) + \frac{b}{\alpha}c_2 \cosh(b\alpha) + \frac{b}{\beta}c_3 \sin(b\beta) - \frac{b}{\beta}c_4 \cos(b\beta) = 0 \tag{133}$$

Fourth boundary condition,

$$\frac{d}{d\xi}\Theta(1) = 0 \tag{134}$$

$$\hat{c}_1 \alpha \cosh(b\alpha) + \hat{c}_2 \alpha \sinh(b\alpha) + \hat{c}_3 \beta \cos(b\beta) - \hat{c}_4 \beta \sin(b\beta) = 0 \tag{135}$$

$$\begin{aligned} &\frac{b}{L\alpha}\left[\alpha^2 + s^2\right]c_1 \alpha \cosh(b\alpha) + \frac{b}{L\alpha}\left[\alpha^2 + s^2\right]c_2 \alpha \sinh(b\alpha) \\ &+ \frac{b}{L\beta}\left[-\beta^2 + s^2\right]c_3 \beta \cos(b\beta) - \frac{b}{L\beta}\left[\beta^2 - s^2\right]c_4 \beta \sin(b\beta) = 0 \end{aligned} \tag{136}$$

$$\begin{aligned} &\frac{b}{\alpha}\left[\alpha^2 + s^2\right]c_1 \alpha \cosh(b\alpha) + \frac{b}{\alpha}\left[\alpha^2 + s^2\right]c_2 \alpha \sinh(b\alpha) \\ &+ \frac{b}{\beta}\left[-\beta^2 + s^2\right]c_3 \beta \cos(b\beta) - \frac{b}{\beta}\left[\beta^2 - s^2\right]c_4 \beta \sin(b\beta) = 0 \end{aligned} \tag{137}$$

Let

$$R_1 = \frac{b}{\alpha}\sinh(b\alpha) \tag{138}$$

$$R_2 = \frac{b}{\alpha}\cosh(b\alpha) \tag{139}$$

$$R_3 = \frac{b}{\beta}\sin(b\beta) \tag{140}$$

$$R_4 = -\frac{b}{\beta}\cos(b\beta) \tag{141}$$

Let

$$R_1' = \left\{\frac{\alpha^2 + s^2}{\alpha}\right\}\{b\alpha \sinh(b\alpha)\} \tag{142}$$

$$R_2' = \left\{\frac{\alpha^2 + s^2}{\alpha}\right\}\{b\alpha \cosh(b\alpha)\} \tag{143}$$

$$R_3' = -\left\{\frac{\beta^2 - s^2}{\beta}\right\}\{\, b\beta \cos(b\beta)\} \tag{144}$$

$$R_4' = \left\{\frac{\beta^2 - s^2}{\beta}\right\}\{- b\beta \sin(b\beta)\} \tag{145}$$

The equations may be expressed in matrix format.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \left(\alpha^2 + s^2\right)/\alpha & 0 & \left(\beta^2 - s^2\right)/\beta \\ R_1 & R_2 & R_3 & R_4 \\ R_1' & R_2' & R_3' & R_4' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{146}$$

The roots are found from the determinant of the coefficient matrix. A numerical method is required.

A simple, trial-and-error method is to substitute candidate values of ω into the variable b. The trial b value is in turn embedded into the system of equations.

References

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3. H. Salarieh and M. Ghorashi, "Free Vibration of Timoshenko Beam with Finite Mass Rigid Tip Load and Flexural-Torsional Coupling," International Journal of Mechanical Sciences 48, 2006.
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APPENDIX A

Example

Consider a fixed-free beam, made from aluminum.

The cross-section is circular with a diameter of 1 inch. The length is 2 inches.

The Bernoulli-Euler natural frequency for pure bending is 6871 Hz, per Reference 4.

The Timoshenko natural frequency for bending, shear, and rotary inertia is 5995 Hz per the method in the main text.

The difference is 15% referenced to the Timoshenko value.