TRANSVERSE VIBRATION OF A BEAM SIMPLY-SUPPORTED AT EACH END WITH BENDING, SHEAR, AND ROTARY INERTIA

By Tom Irvine
Email: tomirvine@aol.com

February 27, 2009

Variables

y(x, t)	=	Transverse displacement
E	=	Elastic modulus
G	=	Shear modulus
I	=	Area moment of inertia
A	=	Cross-section area
r	=	Radius of gyration
k	=	Shear factor
ρ	=	Mass/volume

Equation of Motion

Consider the bending vibration of a beam including both shear and rotary inertia effects. Assume that the beam is uniform. The equations of motion is

$$a^2 \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} - r^2\left[1+\frac{E}{kG}\right]\frac{\partial^4 y}{\partial x^2 \partial t^2} + r^2 \frac{\rho}{kG}\frac{\partial^4 y}{\partial t^4} = 0 \tag{1}$$

where

$$a^2 = \frac{EI}{\rho A} \tag{2}$$

$$r^2 = \frac{I}{A} \tag{3}$$

Assume that the displacement can be separated into spatial and temporal variables.

$$y(x, t) = Y(x)T(t) \tag{4}$$

$$a^2\, Y''''(x)T(t) + Y(x)T''(t) - r^2\left[1+\frac{E}{kG}\right]Y''(x)T''(t) + r^2\frac{\rho}{kG}Y(x)T''''(t) = 0 \tag{5}$$

$$a^2\frac{Y''''(x)}{Y(x)} + \frac{T''(t)}{T(t)} - r^2\left[1+\frac{E}{kG}\right]\frac{Y''(x)T''(t)}{Y(x)T(t)} + r^2\frac{\rho}{kG}\frac{T''''(t)}{T(t)} = 0 \tag{6}$$

Assume that the temporal function can be represented as

$$T(t) = A_n \cos\omega_n t + B_n \sin\omega_n t \tag{7}$$

$$T'(t) = \omega_n\left\{- A_n \sin\omega_n t + B_n \cos\omega_n t\right\} \tag{8}$$

$$T''(t) = \omega_n{}^2\left\{- A_n \cos\omega_n t - B_n \sin\omega_n t\right\} \tag{9}$$

$$T'''(t) = \omega_n{}^3\left\{A_n \sin\omega_n t - B_n \cos\omega_n t\right\} \tag{10}$$

$$T''''(t) = \omega_n{}^4\left\{A_n \cos\omega_n t + B_n \sin\omega_n t\right\} \tag{11}$$

Note that

$$\frac{T''(t)}{T(t)} = -\omega_n{}^2 \tag{12}$$

$$\frac{T''''(t)}{T(t)} = \omega_n{}^4 \tag{13}$$

By substitution,

$$a^2 \frac{Y''''(x)}{Y(x)} - \omega_n^2 + \omega_n^2 r^2 \left[1 + \frac{E}{kG}\right] \frac{Y''(x)}{Y(x)} + \omega_n^4 r^2 \frac{\rho}{kG} = 0 \tag{14}$$

$$a^2 Y''''(x) + \omega_n^2 r^2 \left[1 + \frac{E}{kG}\right] Y''(x) + \omega_n^4 r^2 \frac{\rho}{kG} Y(x) - \omega_n^2 Y(x) = 0 \tag{15}$$

$$a^2 Y''''(x) + \omega_n^2 r^2 \left[1 + \frac{E}{kG}\right] Y''(x) + \omega_n^2 \left[\omega_n^2 r^2 \frac{\rho}{kG} - 1\right] Y(x) = 0 \tag{16}$$

Assume a displacement function

$$Y(x) = a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x) \tag{17}$$

$$\frac{dY(x)}{dx} = a_1 \beta \cosh(\beta x) + a_2 \beta \sinh(\beta x) + a_3 \beta \cos(\beta x) - a_4 \beta \sin(\beta x) \tag{18}$$

$$\frac{d^2 Y(x)}{dx^2} = a_1 \beta^2 \sinh(\beta x) + a_2 \beta^2 \cosh(\beta x) - a_3 \beta^2 \sin(\beta x) - a_4 \beta^2 \cos(\beta x) \tag{19}$$

$$\frac{d^3 Y(x)}{dx^3} = a_1 \beta^3 \cosh(\beta x) + a_2 \beta^3 \sinh(\beta x) - a_3 \beta^3 \cos(\beta x) + a_4 \beta^3 \sin(\beta x) \tag{20}$$

$$\frac{d^4 Y(x)}{dx^4} = a_1 \beta^4 \sinh(\beta x) + a_2 \beta^4 \cosh(\beta x) + a_3 \beta^4 \sin(\beta x) + a_4 \beta^4 \cos(\beta x) \tag{21}$$

APPENDIX A

Simply-Supported

The boundary conditions at the left end x = 0 are

$$Y(0) = 0 \qquad \text{(zero displacement)} \tag{A-1}$$

$$\left.\frac{d^2Y}{dx^2}\right|_{x=0} = 0 \qquad \text{(zero bending moment)} \tag{A-2}$$

The boundary conditions at the free end x = L are

$$Y(L) = 0 \qquad \text{(zero displacement)} \tag{A-3}$$

$$\left.\frac{d^2Y}{dx^2}\right|_{x=L} = 0 \qquad \text{(zero bending moment)} \tag{A-4}$$

Apply boundary condition (A-1) to equation (17).

$$a_2 + a_4 = 0 \tag{A-5}$$

$$a_4 = -a_2 \tag{A-6}$$

Apply boundary condition (A-2) to equation (19).

$$a_2 - a_4 = 0 \tag{A-7}$$

$$a_2 = a_4 \tag{A-8}$$

Equations (A-7) and (A-8) can only be satisfied if

$$a_2 = 0 \tag{A-9}$$

and

$$a_4 = 0 \tag{A-10}$$

The spatial equations thus simplify to

$$Y(x) = a_1 \sinh(\beta x) + a_3 \sin(\beta x) \tag{A-11}$$

$$\frac{d^2\,Y(x)}{dx^2} = a_1\beta^2 \sinh(\beta x) - a_3\beta^2 \sin(\beta x) \tag{A-12}$$

Apply boundary condition (A-6) to equation (17).

$$a_1 \sinh(\beta L) + a_3 \sin(\beta L) = 0 \tag{A-13}$$

Apply boundary condition (A-7) to equation (19).

$$a_1\beta^2 \sinh(\beta L) - a_3\beta^2 \sin(\beta L) = 0 \tag{A-14}$$

$$a_1 \sinh(\beta L) - a_3 \sin(\beta L) = 0 \tag{A-15}$$

$$\begin{bmatrix} \sinh(\beta L) & \sin(\beta L) \\ \sinh(\beta L) & -\sin(\beta L) \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{A-16}$$

By inspection, equation (A-19) can only be satisfied if $a_1 = 0$ and $a_3 = 0$. Set the determinant to zero in order to obtain a nontrivial solution.

$$-\sin(\beta L)\sinh(\beta L) - \sin(\beta L)\sinh(\beta L) = 0 \tag{A-17}$$

$$-2\sin(\beta L)\sinh(\beta L) = 0 \tag{A-18}$$

$$\sin(\beta L)\sinh(\beta L) = 0 \tag{A-19}$$

Equation (A-19) is satisfied if

$$\beta_n L = n\pi, \quad n = 1,2,3,.... \tag{A-20}$$

$$\beta_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, ... \tag{A-21}$$

The displacement function is

$$Y(x) = \hat{a}_1 \{ \sin(\beta L)\sinh(\beta x) + \sinh(\beta L)\sin(\beta x) \} \tag{A-22}$$

$$Y'(x) = \hat{a}_1 \beta \{ \sin(\beta L)\cosh(\beta x) + \sinh(\beta L)\cos(\beta x) \} \tag{A-23}$$

$$Y''(x) = \hat{a}_1 \beta^2 \{ \sin(\beta L)\sinh(\beta x) - \sinh(\beta L)\sin(\beta x) \} \tag{A-24}$$

$$Y'''(x) = \hat{a}_1 \beta^3 \{ \sin(\beta L)\cosh(\beta x) - \sinh(\beta L)\cos(\beta x) \} \tag{A-25}$$

$$Y''''(x) = \hat{a}_1 \beta^4 \{ \sin(\beta L)\sinh(\beta x) + \sinh(\beta L)\sin(\beta x) \} \tag{A-26}$$

$$\begin{aligned} & a^2 \hat{a}_1 \beta^4 \{ \sin(\beta L)\sinh(\beta x) + \sinh(\beta L)\sin(\beta x) \} \\ & + \omega_n{}^2 r^2 \left[1 + \frac{E}{kG} \right] \hat{a}_1 \beta^2 \{ \sin(\beta L)\sinh(\beta x) - \sinh(\beta L)\sin(\beta x) \} \\ & + \omega_n{}^2 \left[\omega_n{}^2 r^2 \frac{\rho}{kG} - 1 \right] \hat{a}_1 \{ \sin(\beta L)\sinh(\beta x) + \sinh(\beta L)\sin(\beta x) \} = 0 \end{aligned} \tag{A-27}$$

$$\begin{aligned} & a^2 \left\{ \frac{n\pi}{L} \right\}^4 \{ \sin(n\pi)\sinh(n\pi x / L) + \sinh(n\pi)\sin(n\pi x / L) \} \\ & + \omega_n{}^2 r^2 \left[1 + \frac{E}{kG} \right] \left\{ \frac{n\pi}{L} \right\}^2 \{ \sin(n\pi)\sinh(n\pi x / L) - \sinh(n\pi)\sin(n\pi x / L) \} \\ & + \omega_n{}^2 \left[\omega_n{}^2 r^2 \frac{\rho}{kG} - 1 \right] \{ \sin(n\pi)\sinh(n\pi x / L) + \sinh(n\pi)\sin(n\pi x / L) \} = 0 \end{aligned} \tag{A-28}$$

$$\begin{aligned} & a^2 \left\{ \frac{n\pi}{L} \right\}^4 \{ \sinh(n\pi)\sin(n\pi x / L) \} + \omega_n{}^2 r^2 \left[1 + \frac{E}{kG} \right] \left\{ \frac{n\pi}{L} \right\}^2 \{ -\sinh(n\pi)\sin(n\pi x / L) \} \\ & + \omega_n{}^2 \left[\omega_n{}^2 r^2 \frac{\rho}{kG} - 1 \right] \{ \sinh(n\pi)\sin(n\pi x / L) \} = 0 \end{aligned} \tag{A-29}$$

$$a^2 \left\{\frac{n\pi}{L}\right\}^4 - \omega_n{}^2 r^2 \left[1+\frac{E}{kG}\right]\left\{\frac{n\pi}{L}\right\}^2 \quad + \quad \omega_n{}^2 \left[\omega_n{}^2 r^2 \frac{\rho}{kG} - 1\right] = 0 \qquad \text{(A-30)}$$

$$a^2 \left\{\frac{n\pi}{L}\right\}^4 - \omega_n{}^2 r^2 \left[1+\frac{E}{kG}\right]\left\{\frac{n\pi}{L}\right\}^2 \quad + \quad \omega_n{}^4 r^2 \frac{\rho}{kG} - \omega_n{}^2 = 0 \qquad \text{(A-31)}$$

$$a^2 \left[\frac{n\pi}{L}\right]^4 - \omega_n{}^2 \left\{1 + r^2 \left[1+\frac{E}{kG}\right]\left[\frac{n\pi}{L}\right]^2\right\} \quad + \quad \omega_n{}^4 r^2 \frac{\rho}{kG} = 0 \qquad \text{(A-32)}$$

Solve for ω_n,

$$\omega_n{}^2 = \left\{\frac{kG}{2r^2\rho}\right\}\left\{1 + r^2\left[1+\frac{E}{kG}\right]\left[\frac{n\pi}{L}\right]^2 \pm \sqrt{\left\{1+r^2\left[1+\frac{E}{kG}\right]\left[\frac{n\pi}{L}\right]^2\right\}^2 - 4r^2\frac{\rho}{kG}a^2\left[\frac{n\pi}{L}\right]^4}\right\}$$

(A-33)

$$\omega_n = \frac{1}{r}\sqrt{\left\{\frac{kG}{2\rho}\right\}\left\{1 + r^2\left[1+\frac{E}{kG}\right]\left[\frac{n\pi}{L}\right]^2 \pm \sqrt{\left\{1+r^2\left[1+\frac{E}{kG}\right]\left[\frac{n\pi}{L}\right]^2\right\}^2 - 4r^2\frac{\rho}{kG}a^2\left[\frac{n\pi}{L}\right]^4}\right\}}$$

(A-34)

Note the shear modulus can be expressed as

$$G = \frac{E}{2(1+\nu)} \qquad \text{(A-35)}$$

where ν is the Poisson's ratio.

Recall

$$a^2 = \frac{EI}{\rho A} \qquad \text{(A-36)}$$

$$r^2 = \frac{I}{A} \qquad \text{(A-37)}$$

References

1. Weaver, Timoshenko, and Young; Vibration Problems in Engineering, Wiley-Interscience, New York, 1990.

2. T. Irvine, Application of the Newton-Raphson Method to Vibration Problems, Vibrationdata Publications, 1999.