Sound Pressure Generated by a Bubble

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This report summarises the analytical expression for the sound pressure generated by a gas bubble in a fluid. The main reference is T.G. Leighton's excellent *The Acoustic Bubble* [1].

1 Pressure Radiated from a Pulsating Sphere

Assume a Newtonian fluid of constant density ρ_0 and sound speed c. A spherical body of rest radius R_0 vibrates radially with wall amplitude U_0, wave number k and angular frequency ω. We have the following analytic expression for the pressure in a fluid at a radius r and time t:

$$P(r,t) = \frac{\rho_0 c R_0 U_0}{r} \cos\chi_0 e^{i(\omega t - k(r-R_0)+\chi_0)} \tag{1}$$

The parameter χ_0 relates the wave number of the vibration to the size of the body and is defined by

$$\cos\chi_0 = \frac{kR_0}{\sqrt{1+(kR_0)^2}}$$

1.1 Long Wavelength Approximation

In the long wavelength limit $kR_0 = \omega_0 R_0/c \ll 1$ we have $\chi_0 \approx \pi/2$ and $e^{i\chi_0} \approx i$. The expression for pressure simplifies considerably:

$$P(r,t) = \frac{i\rho_0 ck{R_0}^2 U_0}{r} e^{\omega t - kr} \tag{2}$$

In section 3.4, the resonant frequency ω_0, and hence the wave number k, is related to the bubble radius. Figure 1 shows the dimensionless parameter kR_0 for a large

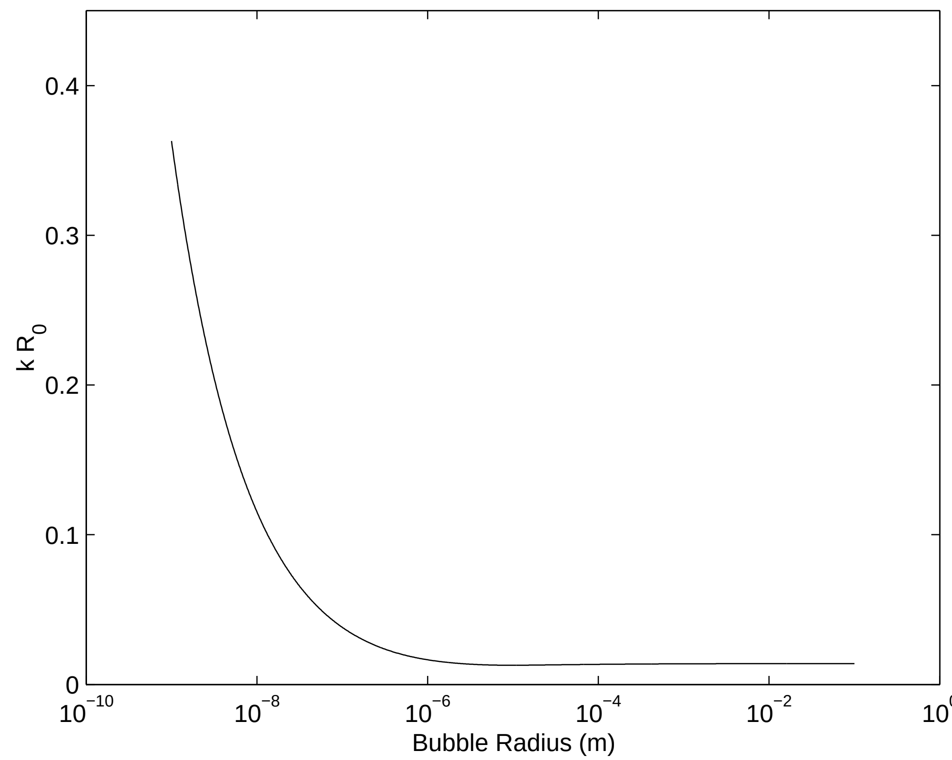

Figure 1: Parameter kR_0

range of bubble radii vibrating at their resonant frequency. Clearly for bubbles larger than 1 μm the long wavelength approximation is valid.

If we are concerned with non-resonant frequencies, then in water we need $kR_0 \equiv \nu R_0/2\pi c \ll 1$, where ν is the linear frequency. At the highest end of human hearing, $\nu \approx 20000$ Hz and so we need $R_0 \ll 0.465$ m, which implies again that the long wavelength approximation is valid.

2 Resonant Frequency of a Bubble

Acoustic vibrations are typically small in amplitude, which leads to linearisation of the standard wave equation and other simplifications. The radial vibration of an gas bubble in a fluid is modeled well as a simple harmonic oscillator whose stiffness and mass parameters relate to the properties of the gas and fluid. In particular the inertial mass of the bubble will relate directly to the work required to move the surrounding fluid. We will take the

2.1 Minnaert Frequency

As a harmonic oscillator, the bubble will have a resonant frequency at which it will vibrate after being subjected to an impulsive force. The resonant frequency is

$$\omega_{0,m} \approx \frac{1}{R_0}\sqrt{\frac{3\kappa p_0}{\rho_0}} \tag{3}$$

or

$$\nu_0 \approx \frac{1}{2\pi R_0}\sqrt{\frac{3\kappa p_0}{\rho_0}} \tag{4}$$

where ν_0 is the linear resonant frequency and p_0 is the static pressure in the fluid. The effects of heat conduction are represented by κ, where $1 \leq \kappa \leq \gamma$ and γ is the ratio of specific heats for the gas. For the isothermal case, $\kappa = 1$, and for the adiabatic case, $\kappa = \gamma$. In most practical cases κ will take on some mid-range value. The effects of surface tension are assumed to be negligible. The classical *Minnaert resonant frequency* assumes that the gas in the bubble is compressed adiabatically, so $\kappa = \gamma$.

3 Damping

We can extend equation 2 to include the effects of damping at the bubble wall. The bubble will lose energy from the radiation of sound, thermal conduction and viscosity. In holding to our harmonic oscillator model, we rewrite the wall amplitude to be $U_0 = U_{0,i}e^{-\beta t}$ where β^{-1} is the time constant of decay of amplitude. In addition, classical harmonic oscillator theory shows that the resonant frequency will be shifted to $\omega_b = \sqrt{\omega_0^2 - \beta^2}$. An examination of the quality factor of the system shows that we can relate β to ω_0, the resonant frequency, by way of a dimensionless damping constant

$$\delta \equiv \frac{2\beta}{\omega_0}$$

Further, it is assumed that radiation, thermal and viscous losses are the only relevant factors in the damping, which contribute linearly:

$$\delta = \delta_{rad} + \delta_{th} + \delta_{vis}$$

Figure 2 shows all three damping coefficients and their sum.

3.1 Radiation Damping

The radiation damping coefficient can be derived by examining the acoustic impedance for spherical waves, which includes the radiation resistance. The radiation damping term is

$$\delta_{rad} = \frac{R_0 \omega_0}{c} \tag{5}$$

3.2 Viscous Damping

The viscous damping coefficient can be derived by examining the Navier-Stokes equations at the bubble wall. While there are no viscous losses inside the bubble, at the wall the bubble loses energy on compression. The viscous damping term is

$$\delta_{vis} = \frac{4\eta}{R_0^2 \rho \omega_0} \tag{6}$$

where η is the coefficient of shear stress in the fluid.

3.3 Thermal Damping

The thermal damping coefficient can be derived using a painful examination of the thermodynamics of the gas inside the bubble. The net result is that if the thermal boundary layer of the bubble is large compared to the radius, then on compression the bubble behaves isothermally and loses energy to the surrounding fluid. If the thermal boundary layer is small compared to the radius, then the bubble behaves adiabatically and no energy is lost through thermal conduction.

The thermal boundary layer thickness is given by $l_D = \sqrt{D_g/2\omega_0}$ where $D_g = K_g/\rho_g C_p$ is the thermal diffusivity of the gas, K_g is the thermal conductivity of the gas, ρ_g is the density of the gas and C_p is specific heat of the gas. The thermal damping coefficient is then given by

$$\delta_{th} = \frac{\frac{\sinh z + \sin z}{\cosh z - \cos z} - \frac{2}{z}}{\frac{z}{3(\gamma - 1)} + \frac{\sinh z - \sin z}{\cosh z - \cos z}} \tag{7}$$

where $z \equiv R_0/l_D$ is the ratio between the bubble radius and the thermal boundary layer thickness.

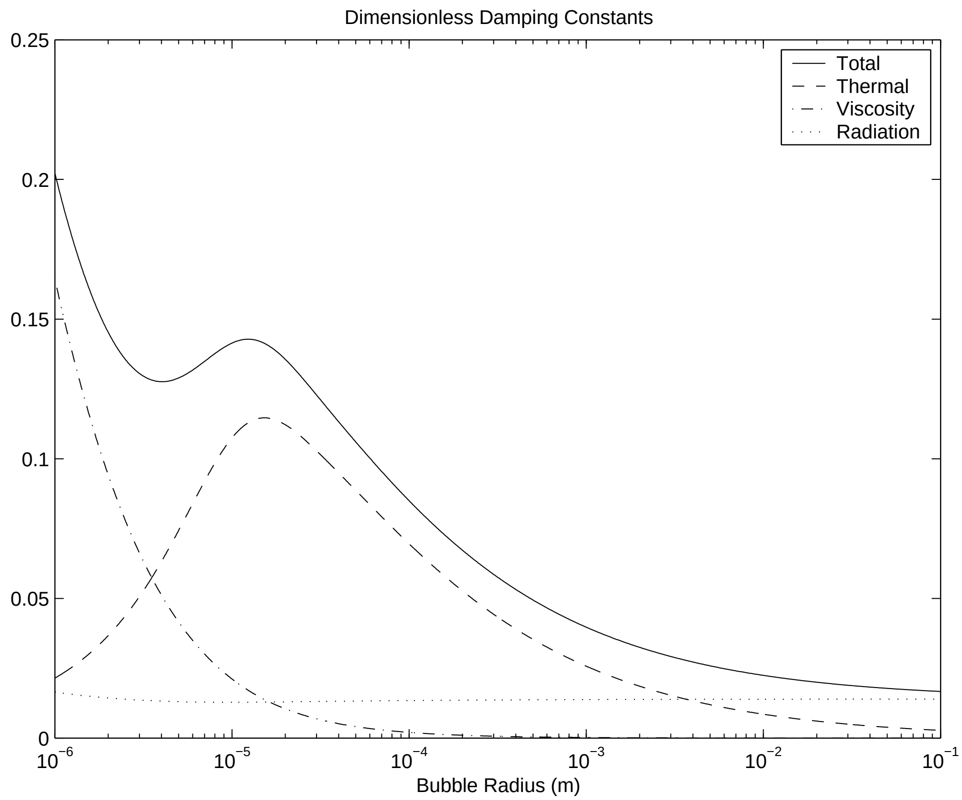

Figure 2: Resonant bubble damping coefficient

3.4 Resonant Frequency with Damping

The effects of damping change the resonant frequency at which the bubble will vibrate. The new frequency is given by

$$\omega_{0,d} = \frac{1}{R_0}\sqrt{\frac{3\gamma p_0}{\rho}}\sqrt{\frac{g}{\alpha}} = \omega_{0,m}\sqrt{\frac{g}{\alpha}} \tag{8}$$

where $\omega_{0,m}$ is the classical Minneart frequency. The effects of thermal conductivity are incorporated into α, a dimensionless constant:

$$\alpha = \left(1+\delta_{th}^2\right)\left[1+\frac{3(\gamma-1)}{z}\left(\frac{\sinh z - \sin z}{\cosh z - \cos z}\right)\right]$$

The effects of the surface tension σ are incorporated into g, a dimensionless constant:

$$g = 1 + \frac{2\sigma}{p_0 R_0} - \frac{2\sigma}{3\kappa p_0 R_0}$$

where $\kappa = \gamma/\alpha$. Figure 3 shows $\sqrt{g/\alpha}$ for various bubble radii and figure 4 compares the damped resonant frequency to the classical Minneart frequency.

4 Calculation of Parameters

The equation for the damped resonant frequency $\omega_{0,d}$ relies on α, which uses the thermal boundary layer l_D, which itself requires the frequency. More explicitly, we have

$$\omega = \omega_{0,m}\sqrt{\frac{g(\omega)}{\alpha(\omega)}}$$

where we have dropped the subscripts on $\omega_{0,d}$. This is a non-linear equation in ω to solve and the solution cannot be written simply. We used Matlab's `fzero` command to solve the equation. For a starting guess we used either the Minneart frequency or if we were calculating the frequencies for a range of radii, the frequency of a nearby radius.

References

[1] T.G. Leighton. *The Acoustic Bubble*. Academic Press, London, 1994.

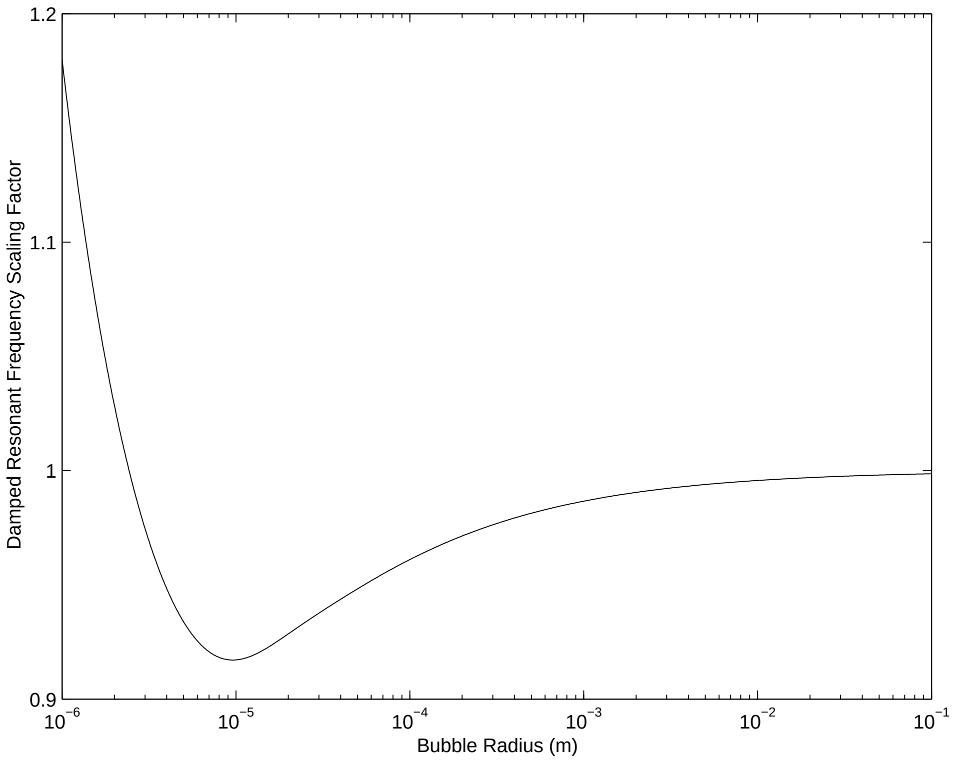

Figure 3: Damped resonant frequency scaling factor

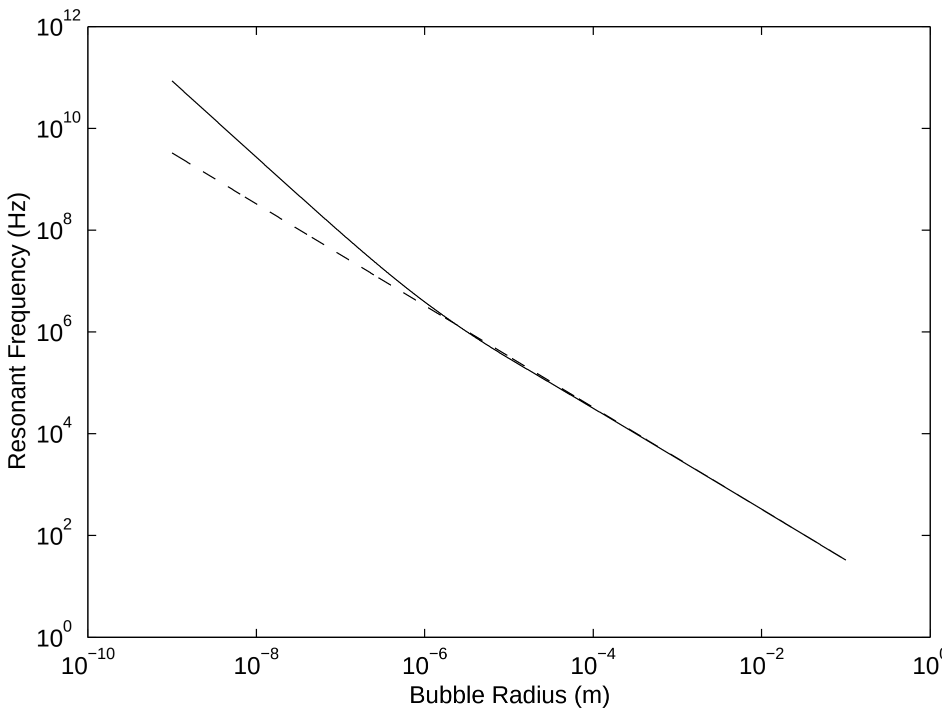

Figure 4: Resonant bubble frequency $\omega_{0,d}/2\pi$