THE RESPONSE OF A CANTILEVER BEAM SUBJECTED TO AN APPLIED CONCENTRATED ARBITRARY FORCE
Revision A

By Tom Irvine
Email: tomirvine@aol.com

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Consider a cantilever beam with an applied force at the free end.

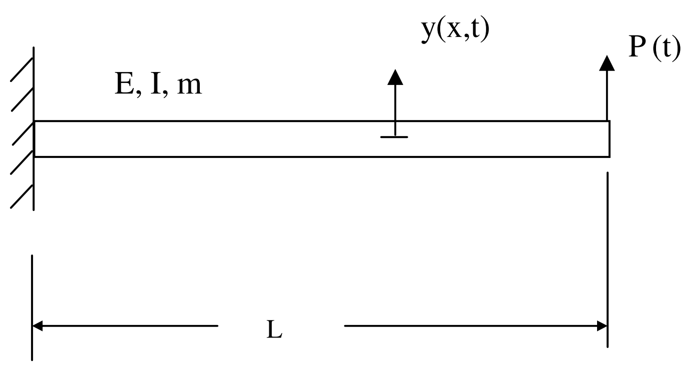


Figure 1.

The variables are

Area moment of inertia	I
Cross-section area	A
Elastic Modulus	E
Length	L
Mass per Volume	ρ
Mass per Length	M
Displacement	u(x,t)
Applied Force	P(t)
Base Excitation Frequency (rad/sec)	ω
Natural Frequency (rad/sec)	ω_n
Viscous Damping Ratio	ξ

Assume uniform mass density and constant cross-section. The governing equation from Reference 1 is

$$-EI\frac{\partial^4 y}{\partial x^4} = m\frac{\partial^2 y}{\partial t^2} \tag{1}$$

The boundary conditions at the fixed end x = 0 for the case without the force are

$$y(0, t) = 0 \qquad \text{(zero displacement)} \tag{2}$$

$$\left.\frac{\partial}{\partial x} y(x,t)\right|_{x=0} = 0 \qquad \text{(zero slope)} \tag{3}$$

The boundary conditions at the free end x = L for the case without the force are

$$\left.\frac{\partial^2}{\partial x^2} y(x,t)\right|_{x=L} = 0 \qquad \text{(zero bending moment)} \tag{4}$$

$$\left.\frac{\partial^3}{\partial x^3} y(x,t)\right|_{x=L} = 0 \qquad \text{(zero shear force)} \tag{5}$$

$$y(x,t) = \sum_{i=1}^{\infty} \phi_i(t)\, Y_i(x) \tag{6}$$

The ϕ_i terms are some unknown functions of time which will be determined by the principle of virtual work.

The natural frequencies are determined from the roots as follows.

Table 1. Roots	
Index	$\beta_n L$
n = 1	1.87510
n = 2	4.69409
n = 3	$5\pi/2$
n = 4	$7\pi/2$

The natural frequency term ω_n is thus

$$\omega_n = \beta_n{}^2 \sqrt{\frac{EI}{m}} \tag{7}$$

The calculation steps are omitted for brevity. The resulting normalized eigenvectors are

$$Y_i(x) = \left\{ \frac{1}{\sqrt{mL}} \right\} \left\{ \left[\cosh(\beta_i x) - \cos(\beta_i x) \right] - D_i \left[\sinh(\beta_i x) - \sin(\beta_i x) \right] \right\} \tag{8}$$

$$D_i = \frac{\cos(\beta_i L) + \cosh(\beta_i L)}{\sin(\beta_i L) + \sinh(\beta_i L)} \tag{9}$$

The virtual transverse displacement δy_i in terms of the mode shapes are

$$\delta y_i = \left\{ \frac{1}{\sqrt{mL}} \right\} \left\{ \left[\cosh(\beta_i x) - \cos(\beta_i x) \right] - D_i \left[\sinh(\beta_i x) - \sin(\beta_i x) \right] \right\} \tag{10}$$

The mass of an element between two adjacent cross sections of the rod is $m\,dx$.

The work δW_I done by inertial forces on the assumed virtual displacement is

$$\delta W_I = \int_0^L (-\,\mathrm{m dx})\, \ddot{y}\, \delta y_i \tag{11}$$

$$\delta y_i = \delta\phi_i\, Y_i \tag{12}$$

$$\delta W_I = -m\, \delta\phi_i \int_0^L \ddot{y}\ Y_i(x)\, dx \tag{13}$$

By substitution,

$$\delta W_I = -m\, \delta\phi_i \int_0^L \left\{ \sum_{j=1}^{\infty} \ddot{\phi}_j(t)\, Y_j(x) \right\} Y_i(x)\, dx \tag{14}$$

$$\delta W_I = -m\ \delta\phi_i \sum_{j=1}^{\infty} \ddot{\phi}_j \int_0^L Y_i(x) Y_j(x)\, dx \tag{15}$$

The orthogonality of the normal mode shapes is such that

$$\int_0^L Y_i(x) Y_j(x)\ dx\ = 0\ , \quad \text{for } i \neq j \tag{16}$$

$$m \int_0^L Y_i(x) Y_j(x)\ dx\ = 1\ , \quad \text{for } i = j \tag{17}$$

$$\delta W_{Ii} = -\ddot{\phi}_i\, \delta\phi_i \tag{18}$$

Now calculate the strain energy U produced by the elastic forces.

$$U = \int_0^L \frac{EI}{2} \left(y''\right)^2 dx \tag{19}$$

$$U = \frac{EI}{2}\int_0^L \left(\sum_{i=1}^{\infty} \phi_i \, Y_i''\right)\left(\sum_{j=1}^{\infty} \phi_j \, Y_j''\right) dx \tag{20}$$

$$U = \frac{EI}{2}\int_0^L \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \phi_i \, \phi_j Y_i'' \, Y_j'' \, dx \tag{21}$$

$$U = \frac{EI}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \phi_i \, \phi_j \int_0^L Y_i'' \, Y_j'' dx \tag{22}$$

The orthogonality relationships are

$$\int_0^L Y_i'' \, Y_j'' dx = 0, \quad \text{for } i \neq j \tag{23}$$

$$\int_0^L Y_i'' \, Y_j'' dx = \beta_i^4, \quad \text{for } i = j \tag{24}$$

Thus,

$$U = \frac{EI}{2}\sum_{i=1}^{\infty} \beta_i^4 \, \phi_i^2 \tag{25}$$

The virtual work of the elastic forces is

$$\delta W_{Ei} = -\frac{\partial U}{\partial \phi_i} \delta\phi_i \tag{26}$$

$$\delta W_{Ei} = -\frac{\partial}{\partial \phi_i}\left\{\frac{EI}{2}\sum_{i=1}^{\infty} \beta_i^{\,4}\, \phi_i^{\,2}\right\}\delta\phi_i \tag{27}$$

$$\delta W_{Ei} = -EI\,\beta_i^{\,4}\,\phi_i\,\delta\phi_i \tag{28}$$

Determine the work of the applied concentrated force.

$$\delta W_{Fi} = P(t)\;\delta y_i(L,t) \tag{29}$$

$$\delta W_{Fi} = P(t)\;Y_i(L)\,\delta\phi_i \tag{30}$$

The total virtual work is thus

$$\ddot{\phi}_i\,\delta\phi_i + EI\,\beta_i^{\,4}\,\phi_i\,\delta\phi_i = P(t)\;Y_i(L)\,\delta\phi_i \tag{31}$$

$$\ddot{\phi}_i + EI\,\beta_i^{\,4}\,\phi_i = P(t)\;Y_i(L) \tag{32}$$

$$\ddot{\phi}_i + \omega_i^{\,2}\,\phi_i = P(t)\;Y_i(L) \tag{33}$$

Add modal damping

$$\ddot{\phi}_i + 2\xi_i\omega_i\,\phi_i + \omega_i^{\,2}\,\phi_i = P(t)\;Y_i(L) \tag{34}$$

<u>Time Domain</u>

The modal equation of motion is

$$\ddot{\phi}_i + 2\,\xi_i\omega_i\,\phi_i + \omega_i^{\,2}\,\phi_i = P(t)Y_i(L) \tag{35}$$

The modal displacement, velocity, and acceleration can then calculated via a numerical method such as the Runge-Kutta or Newmark-Beta method, as given in References 2 and 3, respectively. The Runge-Kutta method, however, may be unstable for stiff systems.

Another method is the ramp invariant digital recursive filtering relationship in Reference 4. This is the method used in the following example.

The physical displacement can then be found via

$$y(x,t) = \sum_{i=1}^{\infty} \phi_i(t)\, Y_i(x) \tag{36}$$

Example

The following analysis is performed via Matlab scripts: cant_beam_force_frf.m & cant_beam_arbit_force.m.

Consider the transverse response of an aluminum, fixed-free, circular rod with the following properties.

Length	L	=	24 inch
Diameter	D	=	1 inch
Area	A	=	0.785 inch^2
Area Moment of Inertia	I	=	0.0491 inch^4
Elastic Modulus	E	=	1.0e+07 lbf/in^2
Mass Density	ρ	=	0.1 lbm/in^3
Speed of Sound in Material	c	=	1.96e+05 in/sec
Viscous Damping Ratio	ξ	=	0.05

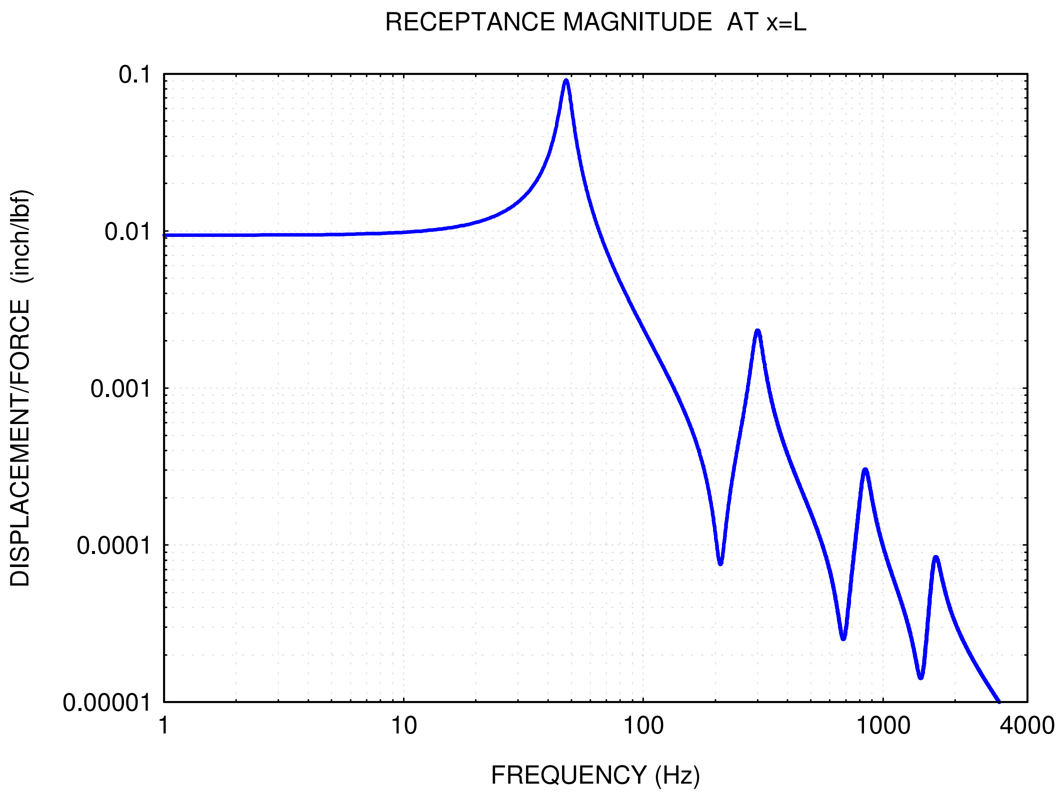


Figure 2.

The resulting transfer functions are shown in Figures 2 through 4.

The first four natural frequencies are

Table 2. Natural Frequencies	
I	f_i (Hz)
1	47.8
2	299
3	837
4	1641

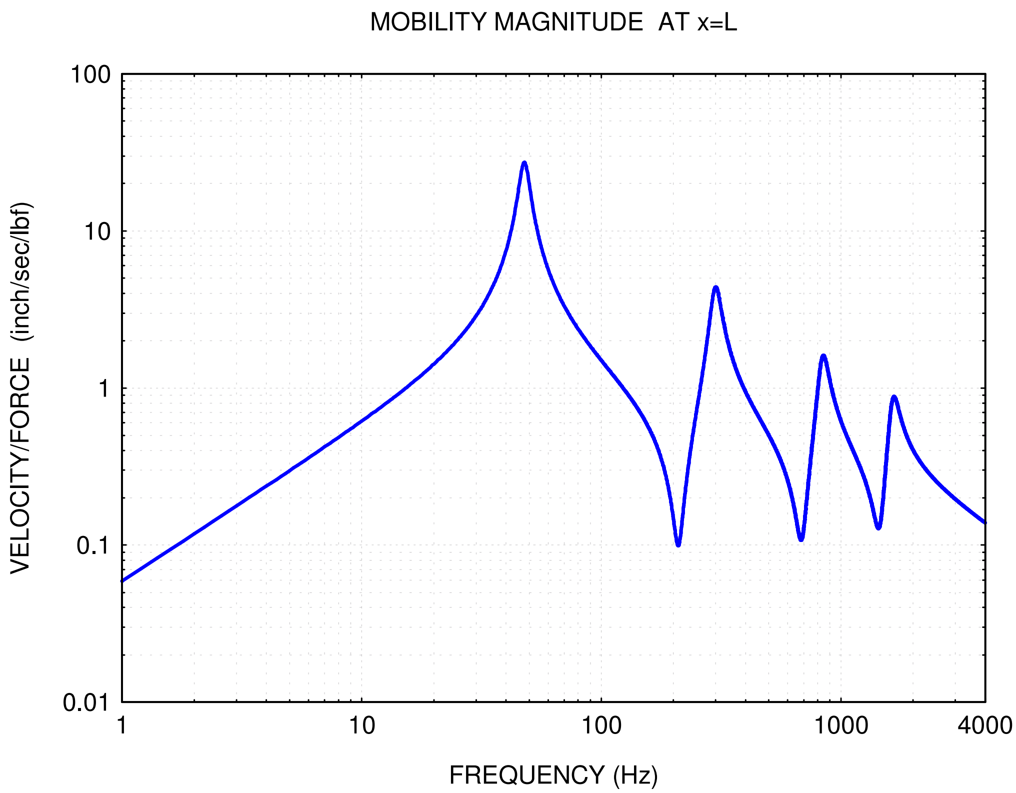

MOBILITY MAGNITUDE AT x=L
VELOCITY/FORCE (inch/sec/lbf)
100
10
1
0.1
0.01
1
10
100
1000
4000
FREQUENCY (Hz)

Figure 3.

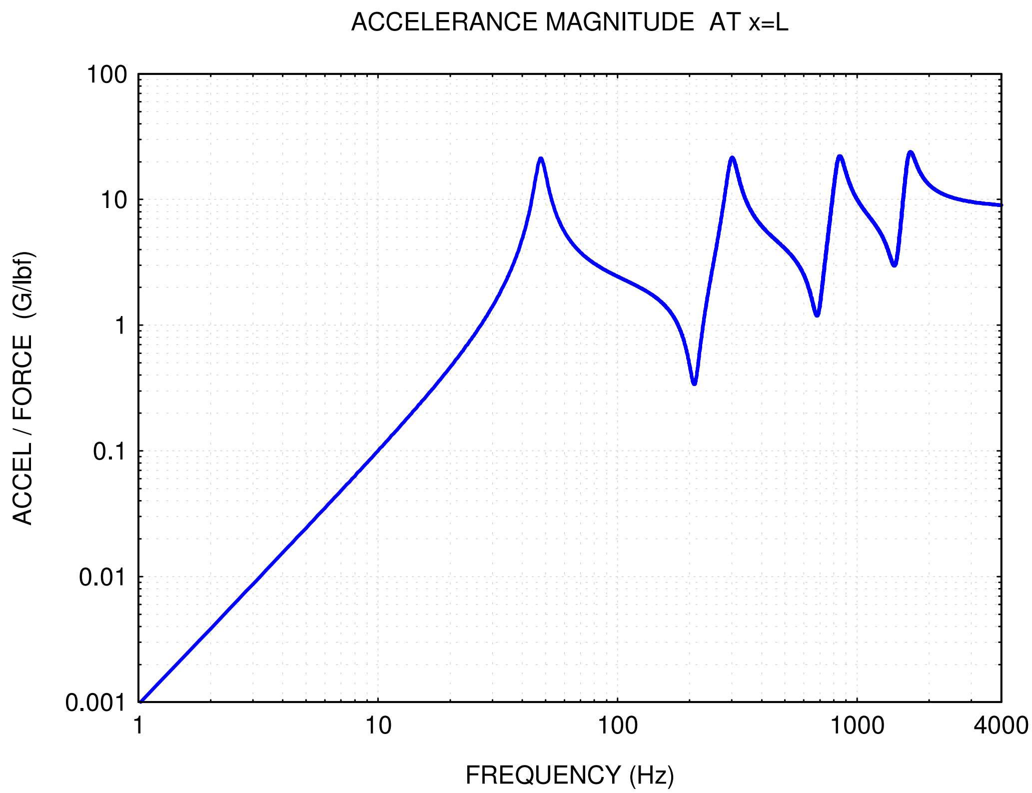
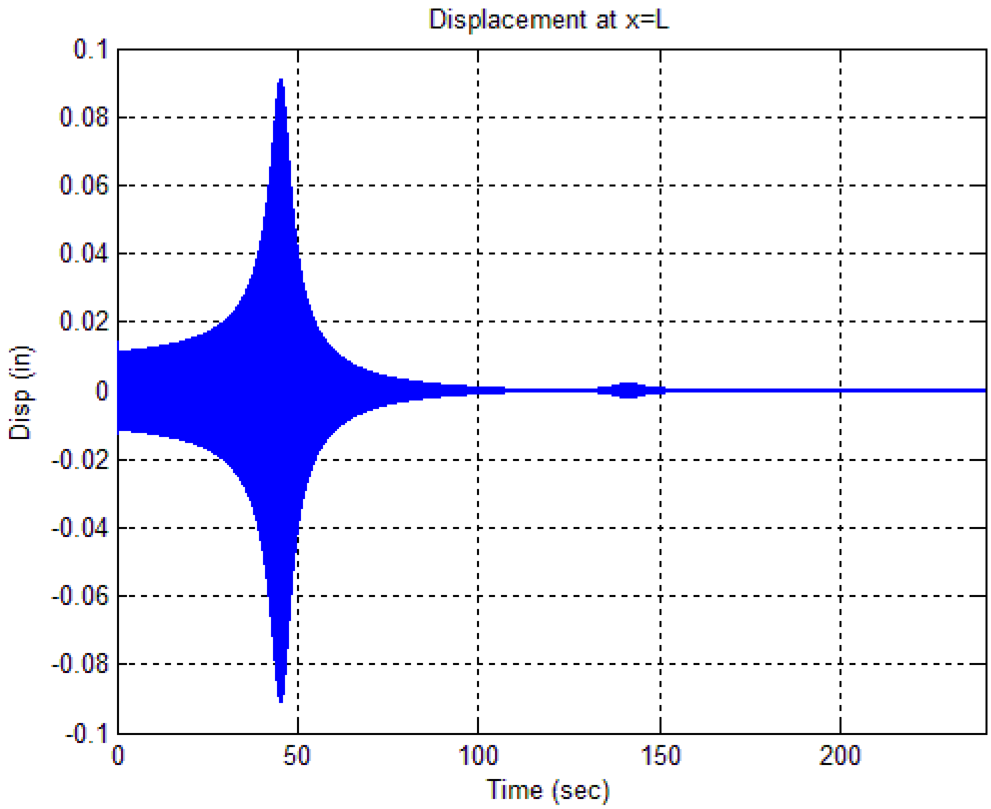


Figure 4.

Now apply a force time history at the free end of the beam. The force is a sine sweep with amplitude of 1 lbf from 20 to 2000 Hz. The sweep rate is logarithmic. The duration is 240 seconds. The results are shown in Figures 5 through 7.

Table 3. Peak Response Values at x=L	
Parameter	Value
Displacement	0.091 in
Velocity	27.5 in/sec
Acceleration	23.6 G



Displacement at x=L
Disp (in)
Time (sec)
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
0
50
100
150
200

Figure 5.

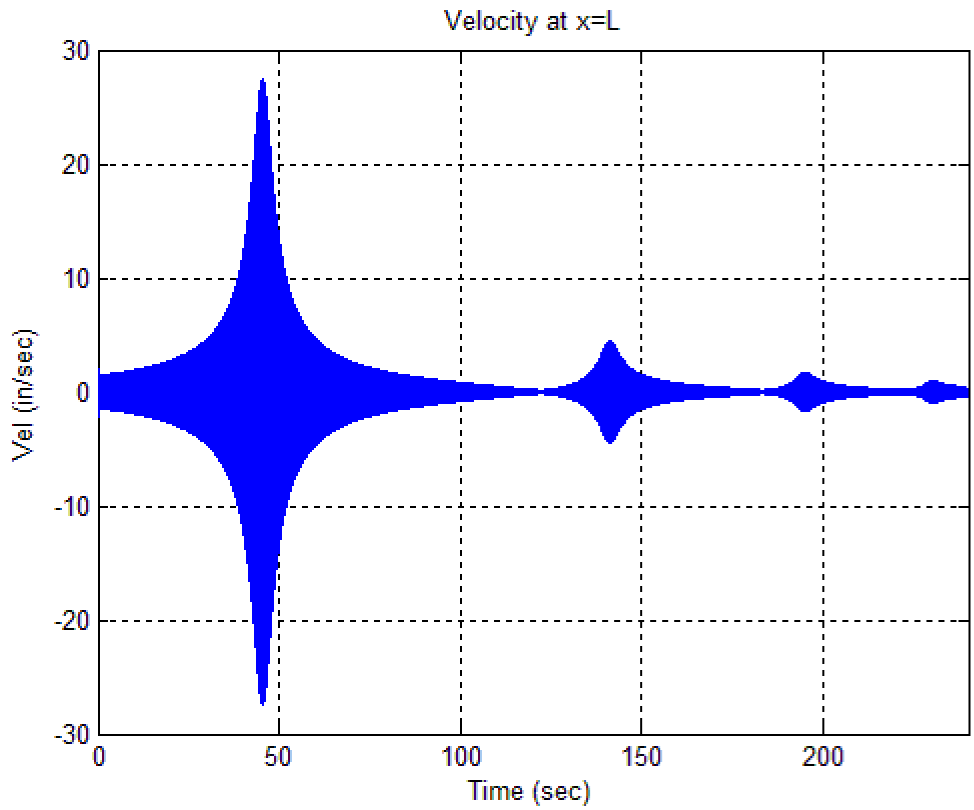

Velocity at x=L
30
20
10
0
-10
-20
-30
Vel (in/sec)
0
50
100
150
200
Time (sec)

Figure 6.

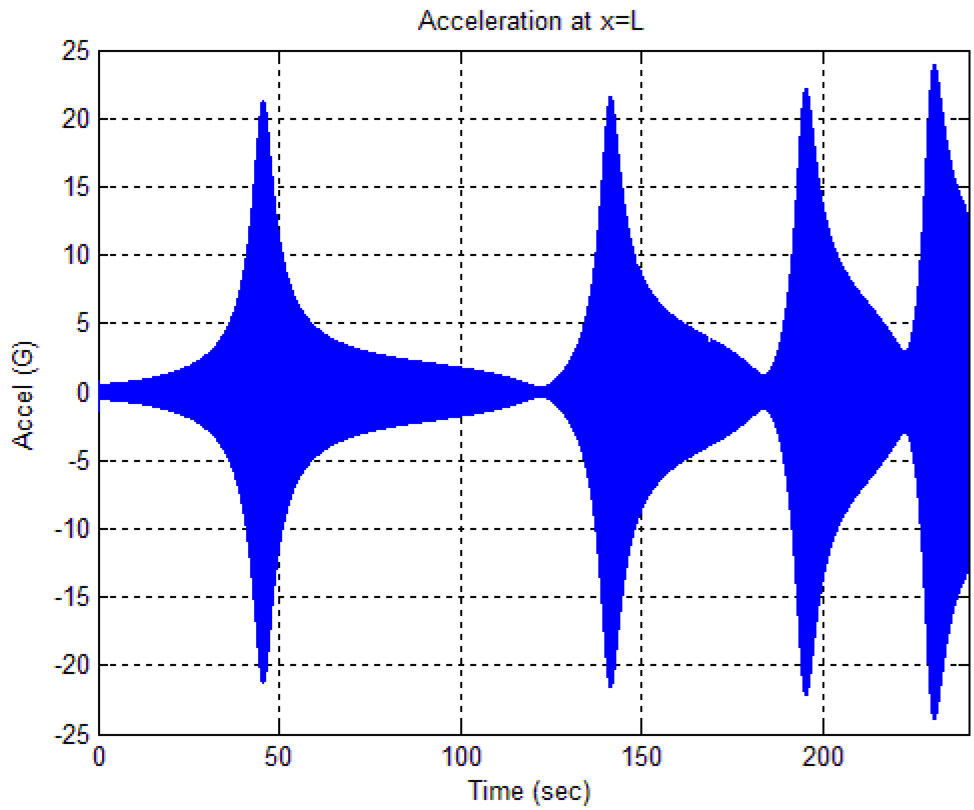


Figure 7.

References

1. Weaver, Timoshenko, and Young; Vibration Problems in Engineering, Wiley Interscience, New York, 1990.

2. W. Thomson, Theory of Vibrations with Applications, Second Edition, Prentice-Hall, New Jersey, 1981.

3. K. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, Englewood Cliffs, New Jersey, 1982.

4. T. Irvine, Modal Transient Analysis of a System Subjected to an Applied Force via a Ramp Invariant Digital Recursive Filtering Relationship, Vibrationdata, 2012.