Consider a cantilever beam with an applied force at the free end.

![Diagram of a cantilever beam](image)

Figure 1.

The variables are

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area moment of inertia</td>
<td>$I$</td>
</tr>
<tr>
<td>Cross-section area</td>
<td>$A$</td>
</tr>
<tr>
<td>Elastic Modulus</td>
<td>$E$</td>
</tr>
<tr>
<td>Length</td>
<td>$L$</td>
</tr>
<tr>
<td>Mass per Volume</td>
<td>$\rho$</td>
</tr>
<tr>
<td>Mass per Length</td>
<td>$m$</td>
</tr>
<tr>
<td>Displacement</td>
<td>$u(x,t)$</td>
</tr>
<tr>
<td>Applied Force</td>
<td>$P(t)$</td>
</tr>
<tr>
<td>Base Excitation Frequency (rad/sec)</td>
<td>$\omega$</td>
</tr>
<tr>
<td>Natural Frequency (rad/sec)</td>
<td>$\omega_n$</td>
</tr>
<tr>
<td>Viscous Damping Ratio</td>
<td>$\xi$</td>
</tr>
</tbody>
</table>
Assume uniform mass density and constant cross-section. The governing equation from Reference 1 is

\[-EI \frac{\partial^4 y}{\partial x^4} = m \frac{\partial^2 y}{\partial t^2}\]  \hspace{1cm} (1)

The boundary conditions at the fixed end \(x = 0\) for the case without the force are

\[y(0, t) = 0 \quad \text{(zero displacement)} \]  \hspace{1cm} (2)

\[\frac{\partial y(x, t)}{\partial x} \bigg|_{x=0} = 0 \quad \text{(zero slope)} \]  \hspace{1cm} (3)

The boundary conditions at the free end \(x = L\) for the case without the force are

\[\frac{\partial^2 y(x, t)}{\partial x^2} \bigg|_{x=L} = 0 \quad \text{(zero bending moment)} \]  \hspace{1cm} (4)

\[\frac{\partial^3 y(x, t)}{\partial x^3} \bigg|_{x=L} = 0 \quad \text{(zero shear force)} \]  \hspace{1cm} (5)

\[y(x, t) = \sum_{i=1}^{\infty} \phi_i(t) Y_i(x) \]  \hspace{1cm} (6)
The $\phi_i$ terms are some unknown functions of time which will be determined by the principle of virtual work.

The natural frequencies are determined from the roots as follows.

<table>
<thead>
<tr>
<th>Index</th>
<th>$\beta_n L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>n = 1</td>
<td>1.87510</td>
</tr>
<tr>
<td>n = 2</td>
<td>4.69409</td>
</tr>
<tr>
<td>n = 3</td>
<td>$5\pi/2$</td>
</tr>
<tr>
<td>n = 4</td>
<td>$7\pi/2$</td>
</tr>
</tbody>
</table>

The natural frequency term $\omega_n$ is thus

$$\omega_n = \beta_n^2 \frac{EI}{m}$$  \hspace{1cm} (7)

The calculation steps are omitted for brevity. The resulting normalized eigenvectors are

$$Y_i(x) = \left\{ \frac{1}{\sqrt{mL}} \right\} \left\{ \cosh(\beta_i x) - \cos(\beta_i x) \right\} - D_i \left[ \sinh(\beta_i x) - \sin(\beta_i x) \right]$$  \hspace{1cm} (8)

$$D_i = \frac{\cos(\beta_i L) + \cosh(\beta_i L)}{\sin(\beta_i L) + \sinh(\beta_i L)}$$  \hspace{1cm} (9)

The virtual transverse displacement $\delta y_i$ in terms of the mode shapes are

$$\delta y_i = \left\{ \frac{1}{\sqrt{mL}} \right\} \left\{ \cosh(\beta_i x) - \cos(\beta_i x) \right\} - D_i \left[ \sinh(\beta_i x) - \sin(\beta_i x) \right]$$  \hspace{1cm} (10)

The mass of an element between two adjacent cross sections of the rod is $m \, dx$. 

3
The work $\delta W_I$ done by inertial forces on the assumed virtual displacement is

$$\delta W_I = \int_0^L (-m \dot{y}) \dot{\delta y}_i$$

(11)

$$\delta y_i = \delta \phi_i Y_i$$

(12)

$$\delta W_I = -m \delta \phi_i \int_0^L \ddot{y} Y_i(x) \text{dx}$$

(13)

By substitution,

$$\delta W_I = -m \delta \phi_i \int_0^L \left\{ \sum_{j=1}^{\infty} \ddot{\phi}_j(t) Y_j(x) \right\} Y_i(x) \text{dx}$$

(14)

$$\delta W_I = -m \delta \phi_i \sum_{j=1}^{\infty} Y_j(x) \int_0^L Y_i(x) Y_j(x) \text{dx}$$

(15)

The orthogonality of the normal mode shapes is such that

$$\int_0^L Y_i(x) Y_j(x) \text{dx} = 0, \quad \text{for } i \neq j$$

(16)

$$m \int_0^L Y_i(x) Y_j(x) \text{dx} = 1, \quad \text{for } i = j$$

(17)

$$\delta W_{Ii} = -\ddot{\phi}_i \delta \phi_i$$

(18)

Now calculate the strain energy $U$ produced by the elastic forces.

$$U = \int_0^L \frac{EI}{2} \left( y'' \right)^2 \text{dx}$$

(19)
The orthogonality relationships are

\[ \int_0^L Y_i'' Y_j'' \, dx = 0, \quad \text{for } i \neq j \]  
\[ \int_0^L Y_i'' Y_j'' \, dx = \beta_i^4, \quad \text{for } i = j \]  

Thus,

\[ U = \frac{EI}{2} \sum_{i=1}^{\infty} \beta_i^4 \phi_i^2 \]  

The virtual work of the elastic forces is

\[ \delta W_{Ei} = -\frac{\partial U}{\partial \phi_i} \delta \phi_i \]
\[
\delta W_{Ei} = -\frac{\partial}{\partial \phi_i} \left\{ \frac{EI}{2} \sum_{i=1}^{\infty} \beta_i^4 \phi_i^2 \right\} \delta \phi_i
\]  
(27)

\[
\delta W_{Ei} = -EI \beta_i^4 \phi_i \delta \phi_i
\]  
(28)

Determine the work of the applied concentrated force.

\[
\delta W_{Fi} = P(t) \delta y_1 (L, t)
\]  
(29)

\[
\delta W_{Fi} = P(t) Y_i (L) \delta \phi_i
\]  
(30)

The total virtual work is thus

\[
\ddot{\phi}_i \delta \phi_i + EI \beta_i^4 \phi_i \delta \phi_i = P(t) Y_i (L) \delta \phi_i
\]  
(31)

\[
\ddot{\phi}_i + EI \beta_i^4 \phi_i = P(t) Y_i (L)
\]  
(32)

\[
\ddot{\phi}_i + \omega_i^2 \phi_i = P(t) Y_i (L)
\]  
(33)

Add modal damping

\[
\ddot{\phi}_i + 2\xi_i \omega_i \phi_i + \omega_i^2 \phi_i = P(t) Y_i (L)
\]  
(34)
**Time Domain**

Let

$$P(t) = \hat{P} \sin(\omega t)$$  \hspace{1cm} (35)

The modal equation of motion is

$$\ddot{\phi}_i + 2 \xi_1 \omega_1 \phi_i + \omega_1^2 \phi_i = P(t)Y_i(L)$$  \hspace{1cm} (36)

$$\ddot{\phi}_i + 2 \xi_1 \omega_1 \phi_i + \omega_1^2 \phi_i = \hat{P}Y_i(L) \sin(\omega t)$$  \hspace{1cm} (37)

Let

$$F_i = \hat{P} Y_i(L)$$  \hspace{1cm} (38)

$$\ddot{\phi}_i + 2 \xi_1 \omega_1 \phi_i + \omega_1^2 \phi_i = F_i \sin(\omega t)$$  \hspace{1cm} (39)

Let

$$\omega_{d,i} = \omega_i \sqrt{1 - \xi_i^2}$$  \hspace{1cm} (40)

The modal displacement is

$$\phi_i(t) =$$

$$\frac{F_i}{(\omega^2 - \omega_i^2)^2 + (2\xi_1 \omega_1)^2} \left\{ 2\xi_1 \omega_1 \omega \cos(\omega t) - \left( \omega^2 - \omega_i^2 \right) \sin(\omega t) \right\}$$

$$+ \frac{1}{\omega_{d,i}} \left\{ \frac{\omega F_i}{(\omega^2 - \omega_i^2)^2 + (2\xi_1 \omega_1)^2} \right\} \left\{ e^{-\xi_1 \omega_1 t} \right\} \left\{ 2\xi_1 \omega_1 \omega_{d,i} \cos(\omega_{d,i} t) \right\}$$

$$+ \frac{1}{\omega_{d,i}} \left\{ \frac{\omega F_i}{(\omega^2 - \omega_i^2)^2 + (2\xi_1 \omega_1)^2} \right\} \left\{ e^{-\xi_1 \omega_1 t} \right\} \left\{ \omega^2 + \omega_i^2 \left[ -1 + 2\xi_i^2 \right] \right\} \sin(\omega_{d,i} t)$$

$$\right\}$$  \hspace{1cm} (41)
The displacement can then be found via

\[ y(x, t) = \sum_{i=1}^{\infty} \phi_i(t) Y_i(x) \quad (42) \]

**Steady-State Frequency Response Function**

Change the forcing function to a harmonic excitation exponential term.

\[ \ddot{\phi}_i + 2\xi_i \omega_i \dot{\phi}_i + \omega_i^2 \phi_i = F_i \exp(j\omega t) \quad (43) \]

\[ \phi_i(t) = \exp(j\omega t) \quad (44) \]

\[ \dot{\phi}_i(t) = j\omega \exp(j\omega t) \quad (45) \]

\[ \ddot{\phi}_i(t) = -\omega^2 \exp(j\omega t) \quad (46) \]

\[ \left\{ -\omega^2 + j\omega 2\xi_i \omega_i + \omega_i^2 \right\} \phi_i \exp(j\omega t) = F_i \exp(j\omega t) \quad (47) \]

\[ \left\{ \left(\omega_i^2 - \omega^2 \right) + j2\xi_i \omega \omega_i \right\} \phi_i = F_i \quad (48) \]

The frequency response function \( H_i(f) \) is thus

\[ \phi_i = \frac{F_i}{(\omega_i^2 - \omega^2) + j2\xi_i \omega \omega_i} \quad (49) \]

\[ \phi_i = \frac{\hat{P} Y_1(L)}{(\omega_i^2 - \omega^2) + j2\xi_i \omega \omega_i} \quad (50) \]
The Fourier transform $\hat{y}(x, f)$ of the displacement response is

$$\hat{y}(x, f) = \sum_{i=1}^{\infty} \phi_i(f) Y_i(x)$$  

(51)

$$\hat{y}(x, \omega) = \sum_{i=1}^{\infty} \frac{P(\omega) Y_i(L) Y_i(x)}{\omega_i^2 - \omega^2 + j 2 \xi_i \omega \omega_i}$$  

(52)

The receptance function $H_r(x, \omega)$ is

$$H_r(x, \omega) = \frac{\hat{y}(x, \omega)}{P(\omega)} = \sum_{i=1}^{\infty} \frac{Y_i(L) Y_i(x)}{\omega_i^2 - \omega^2 + j 2 \xi_i \omega \omega_i}$$  

(53)

where

$P(\omega)$ is the Fourier transform of $P(t) = \hat{P} \sin(\omega t)$

The mobility function $H_m(x, f)$ is

$$H_m(x, \omega) = \frac{\hat{v}(x, \omega)}{P(\omega)} = \sum_{i=1}^{\infty} \frac{j \omega Y_i(L) Y_i(x)}{\omega_i^2 - \omega^2 + j 2 \xi_i \omega \omega_i}$$  

(54)

where

$\hat{v}(x, \omega)$ is the Fourier transform of the velocity
Example, Sinusoidal Excitation

Consider the transverse vibration of an aluminum, fixed-free, circular rod with the following properties.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>L = 24 inch</td>
</tr>
<tr>
<td>Diameter</td>
<td>D = 1 inch</td>
</tr>
<tr>
<td>Area</td>
<td>A = 0.785 inch^2</td>
</tr>
<tr>
<td>Area Moment of Inertia</td>
<td>I = 0.0491 inch^4</td>
</tr>
<tr>
<td>Elastic Modulus</td>
<td>E = 1.0e+07 lbf/in^2</td>
</tr>
<tr>
<td>Mass Density</td>
<td>ρ = 0.1 lbm/in^3</td>
</tr>
<tr>
<td>Speed of Sound in Material</td>
<td>c = 1.96e+05 in/sec</td>
</tr>
<tr>
<td>Viscous Damping Ratio</td>
<td>ξ = 0.05</td>
</tr>
</tbody>
</table>

The analysis is carried out via Matlab script: cant_beam_force_frf.m.
The first four natural frequencies are

<table>
<thead>
<tr>
<th>i</th>
<th>$f_i$ (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>47.8</td>
</tr>
<tr>
<td>2</td>
<td>299</td>
</tr>
<tr>
<td>3</td>
<td>837</td>
</tr>
<tr>
<td>4</td>
<td>1641</td>
</tr>
</tbody>
</table>

The resulting transfer functions are shown in Figures 2 through 4.

Now apply a sinusoidal 1 lbf force at 47.8 Hz at the free end of the beam. Note that the forcing frequency coincides with the fundamental frequency.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>Displacement</td>
<td>0.09125 in</td>
<td>at x=L</td>
</tr>
<tr>
<td>Velocity</td>
<td>27.3 in/sec</td>
<td>at x=L</td>
</tr>
<tr>
<td>Moment</td>
<td>273.2 in-lbf</td>
<td>at x=0</td>
</tr>
<tr>
<td>Stress</td>
<td>2782 psi</td>
<td>at x=0</td>
</tr>
</tbody>
</table>
Figure 2.

RECEPTANCE MAGNITUDE AT x=L

FREQUENCY (Hz)

DISPLACEMENT/FORCE (inch/lbf)
Figure 3.
Figure 4.
Figure 5.
Reference

APPENDIX A

Stress-Velocity Relationship

The bending moment is

\[
M(x, \omega) = EI \sum_{i=1}^{\infty} \frac{P(\omega) Y_i(L) Y_i''(x)}{(\omega_i^2 - \omega^2) + j 2 \xi_i \omega \omega_i} \quad (A-1)
\]

The bending stress \( \sigma(x, \omega) \) is

\[
\sigma(x, \omega) = \frac{\hat{c}}{I} M(x, \omega) \quad (A-2)
\]

where

\[
\hat{c} = \text{Distance to neutral axis}
\]

\[
\sigma(x, \omega) = \hat{c} E \sum_{i=1}^{\infty} \frac{P(\omega) Y_i(L) Y_i''(x)}{(\omega_i^2 - \omega^2) + j 2 \xi_i \omega \omega_i} \quad (A-3)
\]

The bending stress for the first mode at the fixed end is

\[
\sigma(0, \omega) = \hat{c} E \frac{P(\omega) Y_1(L) Y_1''(0)}{(\omega_1^2 - \omega^2) + j 2 \xi_1 \omega \omega_1} \quad (A-4)
\]

The velocity is

\[
\dot{v}(x, \omega) = \sum_{i=1}^{\infty} \frac{j \omega P(\omega) Y_i(L) Y_i(x)}{(\omega_i^2 - \omega^2) + j 2 \xi_i \omega \omega_i} \quad (A-5)
\]
The velocity for the first mode at the free end is

$$\dot{v}(L, \omega) = \frac{j \omega P(\omega) Y_1(L) Y_1(L)}{(\omega_1^2 - \omega^2) + j 2 \xi_1 \omega \omega_1} \quad (A-6)$$

$$\sigma(0, \omega) = \dot{c} E \frac{P(\omega) Y_1(L) Y_1''(0)}{(\omega_1^2 - \omega^2) + j 2 \xi_1 \omega \omega_1} \quad (A-7)$$

The second derivative of the mode shape is

$$Y_1''(x) = \left\{ \frac{\beta_i^2}{\sqrt{mL}} \right\}\left\{ [\cosh(\beta_i x) + \cos(\beta_i x)] - D_i [\sinh(\beta_i x) + \sin(\beta_i x)] \right\} \quad (A-8)$$

$$Y_1''(0) = \frac{2 \beta_i^2}{\sqrt{mL}} \quad (A-9)$$

$$\sigma(0, \omega) = \dot{c} E \frac{2 \beta_1^2}{\sqrt{mL}} \frac{P(\omega) Y_1(L)}{(\omega_1^2 - \omega^2) + j 2 \xi_1 \omega \omega_1} \quad (A-10)$$

$$\omega_n = \beta_n^2 \sqrt{\frac{EI}{m}} \quad (A-11)$$

$$\beta_n^2 = \omega_n \sqrt{\frac{m}{EI}} \quad (A-12)$$
\[
\sigma(0, \omega) = \hat{c} E \frac{2}{\sqrt{mL}} \omega_1 \sqrt{\frac{m}{EI}} \frac{P(\omega) Y_1(L)}{(\omega_1^2 - \omega^2) + j 2\xi_1 \omega \omega_1}
\] (A-13)

\[
\sigma(0, \omega) = 2\hat{c} \omega_1 \sqrt{\frac{E}{IL}} \frac{P(\omega) Y_1(L)}{(\omega_1^2 - \omega^2) + j 2\xi_1 \omega \omega_1}
\] (A-14)

Again,

\[
\hat{v}(L, \omega) = -\frac{j\omega P(\omega) Y_1(L) Y_1(L)}{(\omega_1^2 - \omega^2) + j 2\xi_1 \omega \omega_1}
\] (A-15)

The bending stress for the first mode at the fixed end is

\[
\sigma(0, \omega) = 2\hat{c} \frac{\omega_1}{j\omega} \sqrt{\frac{E}{IL}} \frac{\hat{v}(L, \omega)}{Y_1(L)}
\] (A-16)

The magnitude is

\[
|\sigma(0, \omega)| = 2\hat{c} \frac{\omega_1}{\omega} \sqrt{\frac{E}{IL}} \left|\frac{\hat{v}(L, \omega)}{Y_1(L)}\right|
\] (A-17)