THE TRANSVERSE VIBRATION RESPONSE OF A
CANTILEVER BEAM SUBJECTED TO AN APPLIED CONCENTRATED FORCE
Revision C

By Tom Irvine
Email: tom@vibrationdata.com

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Consider a cantilever beam with an applied force at the free end.

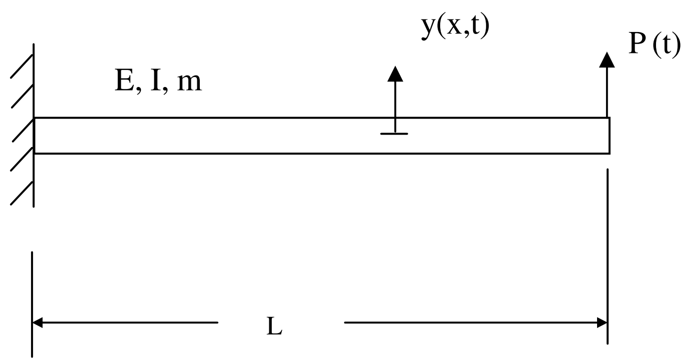

Figure 1.

The variables are

Area moment of inertia	I
Cross-section area	A
Elastic Modulus	E
Length	L
Mass per Volume	ρ
Mass per Length	m
Displacement	u(x,t)
Applied Force	P(t)
Base Excitation Frequency (rad/sec)	ω
Natural Frequency (rad/sec)	ω_n
Viscous Damping Ratio	ξ

Assume uniform mass density and constant cross-section. The governing equation from Reference 1 is

$$-EI\frac{\partial^4 y}{\partial x^4} = m\frac{\partial^2 y}{\partial t^2} \tag{1}$$

The boundary conditions at the fixed end x = 0 for the case without the force are

$$y(0, t) = 0 \qquad \text{(zero displacement)} \tag{2}$$

$$\left.\frac{\partial}{\partial x}y(x,t)\right|_{x=0} = 0 \qquad \text{(zero slope)} \tag{3}$$

The boundary conditions at the free end x = L for the case without the force are

$$\left.\frac{\partial^2}{\partial x^2}y(x,t)\right|_{x=L} = 0 \qquad \text{(zero bending moment)} \tag{4}$$

$$\left.\frac{\partial^3}{\partial x^3}y(x,t)\right|_{x=L} = 0 \qquad \text{(zero shear force)} \tag{5}$$

$$y(x,t) = \sum_{i=1}^{\infty} \phi_i(t)\, Y_i(x) \tag{6}$$

The ϕ_i terms are some unknown functions of time which will be determined by the principle of virtual work.

The natural frequencies are determined from the roots as follows.

Table 1. Roots	
Index	$\beta_n L$
n = 1	1.87510
n = 2	4.69409
n = 3	$5\pi/2$
n = 4	$7\pi/2$

The natural frequency term ω_n is thus

$$\omega_n = \beta_n^{\,2}\sqrt{\frac{EI}{m}} \tag{7}$$

The calculation steps are omitted for brevity. The resulting normalized eigenvectors are

$$Y_i(x) = \left\{\frac{1}{\sqrt{mL}}\right\}\left\{\left[\cosh(\beta_i x) - \cos(\beta_i x)\right] - D_i\left[\sinh(\beta_i x) - \sin(\beta_i x)\right]\right\} \tag{8}$$

$$D_i = \frac{\cos(\beta_i L) + \cosh(\beta_i L)}{\sin(\beta_i L) + \sinh(\beta_i L)} \tag{9}$$

The virtual transverse displacement δy_i in terms of the mode shapes are

$$\delta y_i = \left\{\frac{1}{\sqrt{mL}}\right\}\left\{\left[\cosh(\beta_i x) - \cos(\beta_i x)\right] - D_i\left[\sinh(\beta_i x) - \sin(\beta_i x)\right]\right\} \tag{10}$$

The mass of an element between two adjacent cross sections of the rod is $m\,dx$.

The work δW_I done by inertial forces on the assumed virtual displacement is

$$\delta W_I = \int_0^L (-\,m dx)\,\ddot{y}\,\delta y_i \tag{11}$$

$$\delta y_i = \delta\phi_i\, Y_i \tag{12}$$

$$\delta W_I = -m\,\delta\phi_i \int_0^L \ddot{y}\; Y_i(x)\,dx \tag{13}$$

By substitution,

$$\delta W_I = -m\,\delta\phi_i \int_0^L \left\{ \sum_{j=1}^{\infty} \ddot{\phi}_j(t)\, Y_j(x) \right\} Y_i(x)\,dx \tag{14}$$

$$\delta W_I = -m\;\delta\phi_i \sum_{j=1}^{\infty} \ddot{\phi}_j \int_0^L Y_i(x) Y_j(x)\,dx \tag{15}$$

The orthogonality of the normal mode shapes is such that

$$\int_0^L Y_i(x) Y_j(x)\; dx \;= 0\,, \quad \text{for } i \neq j \tag{16}$$

$$m\int_0^L Y_i(x) Y_j(x)\; dx \;= 1\,, \quad \text{for } i = j \tag{17}$$

$$\delta W_{Ii} = -\ddot{\phi}_i\,\delta\phi_i \tag{18}$$

Now calculate the strain energy U produced by the elastic forces.

$$U = \int_0^L \frac{EI}{2}\left(y''\right)^2 dx \tag{19}$$

$$U = \frac{EI}{2}\int_0^L \left(\sum_{i=1}^{\infty} \phi_i \, Y_i''\right)\left(\sum_{j=1}^{\infty} \phi_j \, Y_j''\right)dx \tag{20}$$

$$U = \frac{EI}{2}\int_0^L \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\phi_i \, \phi_j Y_i'' \, Y_j'' \, dx \tag{21}$$

$$U = \frac{EI}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\phi_i \, \phi_j \int_0^L Y_i'' \, Y_j'' dx \tag{22}$$

The orthogonality relationships are

$$\int_0^L Y_i'' \, Y_j'' dx = 0, \quad \text{for } i \neq j \tag{23}$$

$$\int_0^L Y_i'' \, Y_j'' dx = \beta_i^4, \quad \text{for } i = j \tag{24}$$

Thus,

$$U = \frac{EI}{2}\sum_{i=1}^{\infty} \beta_i^4 \, \phi_i^2 \tag{25}$$

The virtual work of the elastic forces is

$$\delta W_{Ei} = -\frac{\partial U}{\partial \phi_i}\delta\phi_i \tag{26}$$

$$\delta W_{Ei} = -\frac{\partial}{\partial \phi_i}\left\{\frac{EI}{2}\sum_{i=1}^{\infty}\beta_i^{4}\,\phi_i^{2}\right\}\delta\phi_i \tag{27}$$

$$\delta W_{Ei} = -EI\beta_i^{4}\,\phi_i\,\delta\phi_i \tag{28}$$

Determine the work of the applied concentrated force.

$$\delta W_{Fi} = P(t)\;\delta y_i\,(L,t) \tag{29}$$

$$\delta W_{Fi} = P(t)\;Y_i\,(L)\,\delta\phi_i \tag{30}$$

The total virtual work is thus

$$\ddot{\phi}_i\;\delta\phi_i + EI\beta_i^{4}\,\phi_i\,\delta\phi_i = P(t)\;Y_i\,(L)\,\delta\phi_i \tag{31}$$

$$\ddot{\phi}_i + EI\beta_i^{4}\,\phi_i = P(t)\;Y_i\,(L) \tag{32}$$

$$\ddot{\phi}_i + \omega_i^{2}\,\phi_i = P(t)\;Y_i\,(L) \tag{33}$$

Add modal damping

$$\ddot{\phi}_i + 2\xi_i\omega_i\,\phi_i + \omega_i^{2}\,\phi_i = P(t)\;Y_i\,(L) \tag{34}$$

Time Domain

Let

$$P(t) = \hat{P}\sin(\omega t) \tag{35}$$

The modal equation of motion is

$$\ddot{\phi}_i + 2\xi_i\omega_i\dot{\phi}_i + \omega_i^2\phi_i = P(t)Y_i(L) \tag{36}$$

$$\ddot{\phi}_i + 2\xi_i\omega_i\dot{\phi}_i + \omega_i^2\phi_i = \hat{P}Y_i(L)\sin(\omega t) \tag{37}$$

Let

$$F_i = \hat{P}\,Y_i(L) \tag{38}$$

$$\ddot{\phi}_i + 2\xi_i\omega_i\dot{\phi}_i + \omega_i^2\phi_i = F_i\,\sin(\omega t) \tag{39}$$

Let

$$\omega_{d,i} = \omega_i\sqrt{1-\xi_i^{\,2}} \tag{40}$$

The modal displacement is

$$\begin{aligned}
\phi_i(t) = \\
&+\frac{F_i}{\left(\omega^2-\omega_i^2\right)^2+\left(2\xi_i\,\omega\omega_i\right)^2}\left\{2\xi_i\omega_i\omega\cos(\omega t)-\left(\omega^2-\omega_i^2\right)\sin(\omega t)\right\} \\
&+\frac{1}{\omega_{d,i}}\left\{\frac{\omega F_i}{\left(\omega^2-\omega_i^2\right)^2+\left(2\xi_i\,\omega\omega_i\right)^2}\right\}\left\{e^{-\xi_i\omega_i t}\right\}\left\{2\xi_i\omega_i\omega_{d,i}\cos(\omega_{d,i}t)\right\} \\
&+\frac{1}{\omega_{d,i}}\left\{\frac{\omega F_i}{\left(\omega^2-\omega_i^2\right)^2+\left(2\xi_i\,\omega\omega_i\right)^2}\right\}\left\{e^{-\xi_i\omega_i t}\right\}\left\{\left[\omega^2+\omega_i^2\left[-1+2\xi_i^{\,2}\right]\right]\sin(\omega_{d,i}t)\right\}
\end{aligned} \tag{41}$$

The displacement can then be found via

$$y(x,t) = \sum_{i=1}^{\infty} \phi_i(t)\, Y_i(x) \tag{42}$$

Steady-State Frequency Response Function

Change the forcing function to a harmonic excitation exponential term.

$$\ddot{\phi}_i + 2\xi_i\omega_i\, \dot{\phi}_i + \omega_i^{\,2}\, \phi_i = F_i \exp(j\omega t) \tag{43}$$

$$\phi_i(t) = \exp(j\omega t) \tag{44}$$

$$\dot{\phi}_i(t) = j\omega \exp(j\omega t) \tag{45}$$

$$\ddot{\phi}_i(t) = -\omega^2 \exp(j\omega t) \tag{46}$$

$$\left\{ -\omega^2 + j\omega\, 2\xi_i\omega_i + \omega_i^{\,2} \right\} \phi_i \exp(j\omega t) = F_i \exp(j\omega t) \tag{47}$$

$$\left\{ \left(\omega_i^{\,2} - \omega^2\right) + j\, 2\xi_i\omega\, \omega_i \right\} \phi_i = F_i \tag{48}$$

The frequency response function $H_i(f)$ is thus

$$\phi_i = \frac{F_i}{(\omega_i^{\,2} - \omega^2) + j\, 2\xi_i\omega\, \omega_i} \tag{49}$$

$$\phi_i = \frac{\hat{P}\, Y_i(L)}{(\omega_i^{\,2} - \omega^2) + j\, 2\xi_i\omega\, \omega_i} \tag{50}$$

The Fourier transform $\hat{y}(x,f)$ of the displacement response is

$$\hat{y}(x,f) = \sum_{i=1}^{\infty} \phi_i(f)\, Y_i(x) \tag{51}$$

$$\hat{y}(x,\omega) = \sum_{i=1}^{\infty} \frac{P(\omega)\; Y_i(L)\, Y_i(x)}{(\omega_i{}^2 - \omega^2) + j\, 2\xi_i \omega\, \omega_i} \tag{52}$$

The receptance function $H_r(x,\omega)$ is

$$H_r(x,\omega) = \frac{\hat{y}(x,\omega)}{P(\omega)} = \sum_{i=1}^{\infty} \frac{Y_i(L)\, Y_i(x)}{(\omega_i{}^2 - \omega^2) + j\, 2\xi_i \omega\, \omega_i} \tag{53}$$

where

$P(\omega)$ is the Fourier transform of $P(t) = \hat{P}\sin(\omega t)$

The mobility function $H_m(x,f)$ is

$$H_m(x,\omega) = \frac{\hat{v}(x,\omega)}{P(\omega)} = \sum_{i=1}^{\infty} \frac{j\omega\, Y_i(L)\, Y_i(x)}{(\omega_i{}^2 - \omega^2) + j\, 2\xi_i \omega\, \omega_i} \tag{54}$$

where

$\hat{v}(x,\omega)$ is the Fourier transform of the velocity

Example, Sinusoidal Excitation

Consider the transverse vibration of an aluminum, fixed-free, circular rod with the following properties.

Length	L	=	24 inch
Diameter	D	=	1 inch
Area	A	=	0.785 inch^2
Area Moment of Inertia	I	=	0.0491 inch^4
Elastic Modulus	E	=	1.0e+07 lbf/in^2
Mass Density	ρ	=	0.1 lbm/in^3
Speed of Sound in Material	c	=	1.96e+05 in/sec
Viscous Damping Ratio	ξ	=	0.05

The analysis is carried out via Matlab script: cant_beam_force_frf.m.

The first four natural frequencies are

Table 2. Natural Frequencies	
i	f_i (Hz)
1	47.8
2	299
3	837
4	1641

The resulting transfer functions are shown in Figures 2 through 4.

Now apply a sinusoidal 1 lbf force at 47.8 Hz at the free end of the beam. Note that the forcing frequency coincides with the fundamental frequency.

Table 3. Peak Response Values		
Parameter	Value	Location
Displacement	0.09125 in	at x=L
Velocity	27.3 in/sec	at x=L
Moment	273.2 in-lbf	at x=0
Stress	2782 psi	at x=0

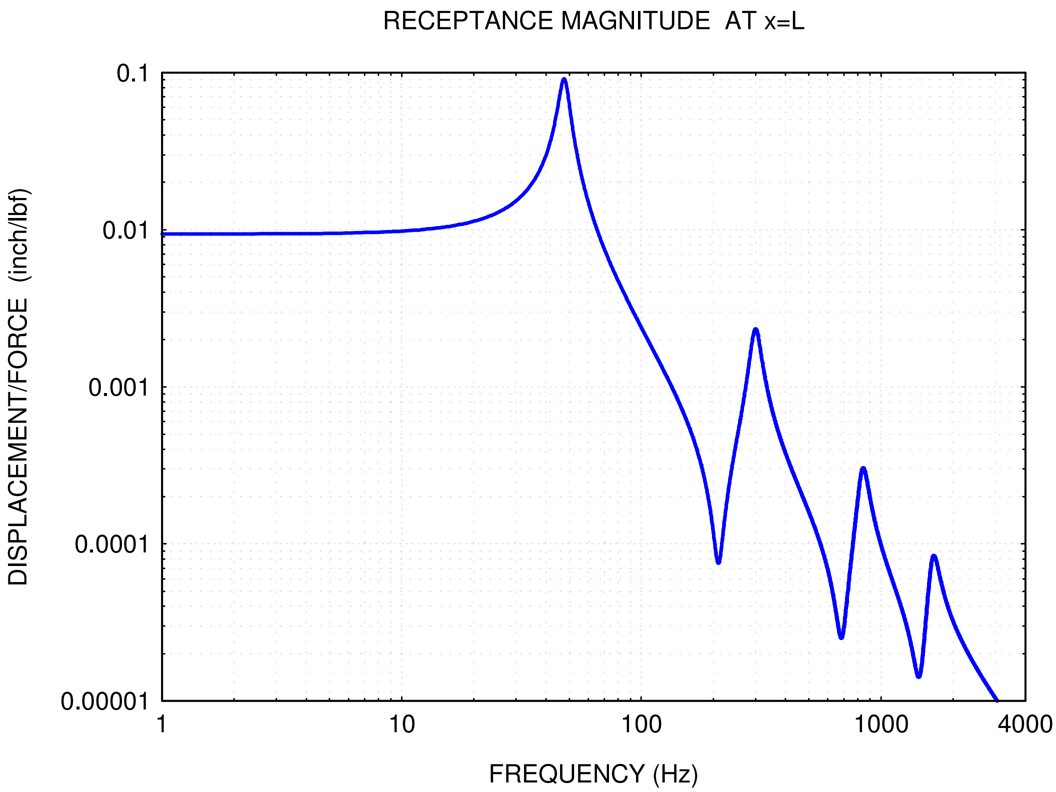
RECEPTANCE MAGNITUDE AT x=L
DISPLACEMENT/FORCE (inch/lbf)
0.1
0.01
0.001
0.0001
0.00001
1
10
100
1000
4000
FREQUENCY (Hz)

Figure 2.

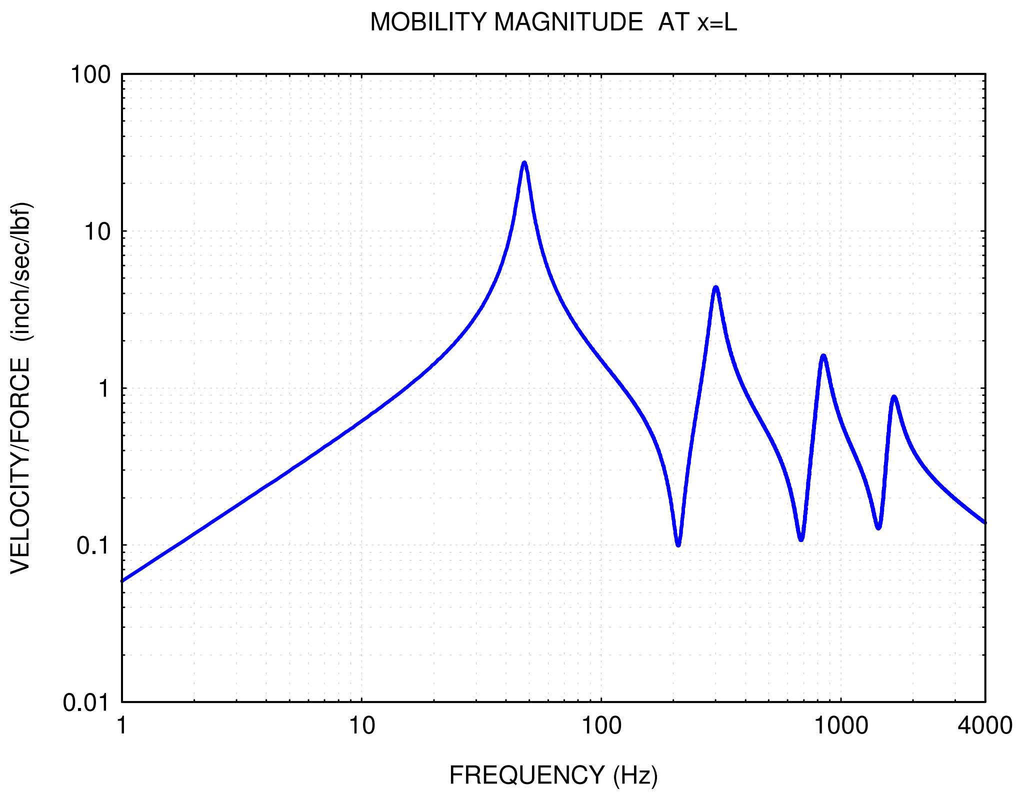
MOBILITY MAGNITUDE AT x=L
VELOCITY/FORCE (inch/sec/lbf)
100
10
1
0.1
0.01
1
10
100
1000
4000
FREQUENCY (Hz)

Figure 3.

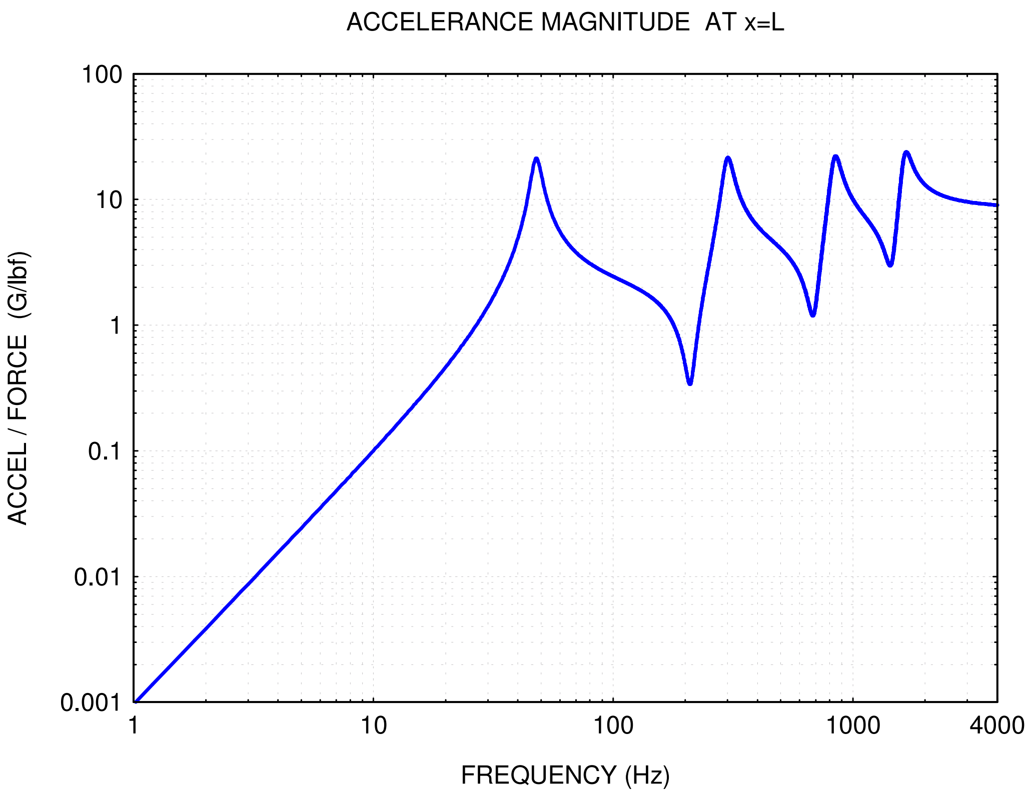
ACCELERANCE MAGNITUDE AT x=L
ACCEL / FORCE (G/lbf)
100
10
1
0.1
0.01
0.001
1
10
100
1000
4000
FREQUENCY (Hz)

Figure 4.

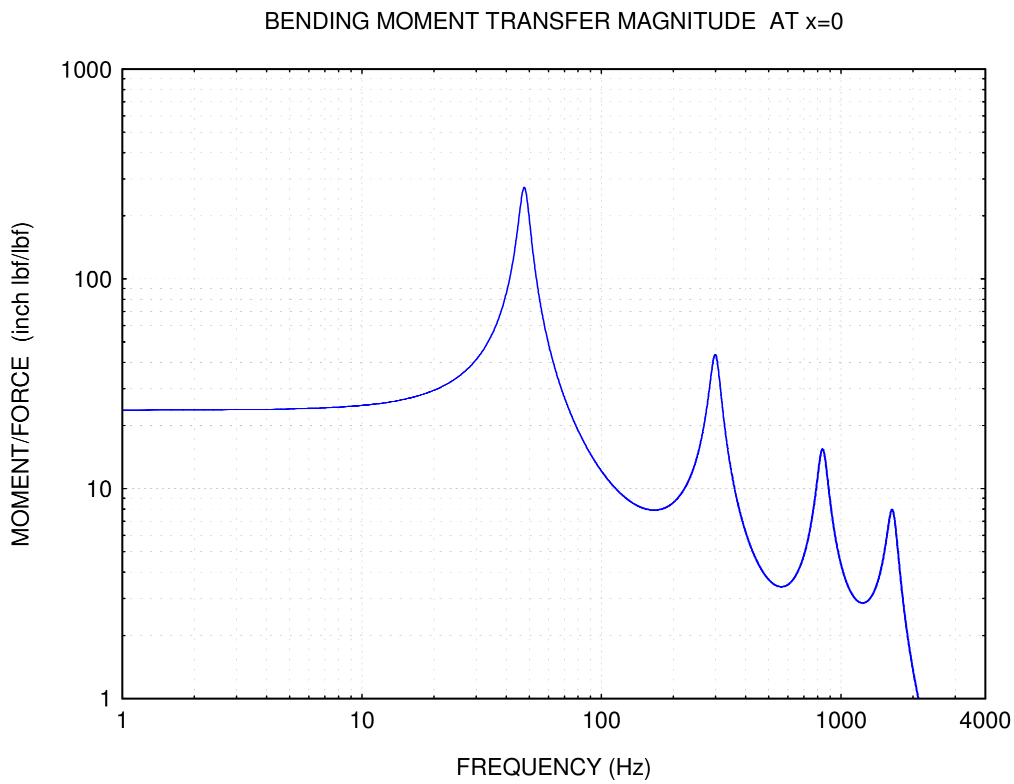
BENDING MOMENT TRANSFER MAGNITUDE AT x=0
MOMENT/FORCE (inch lbf/lbf)
1000
100
10
1
1
10
100
1000
4000
FREQUENCY (Hz)

Figure 5.

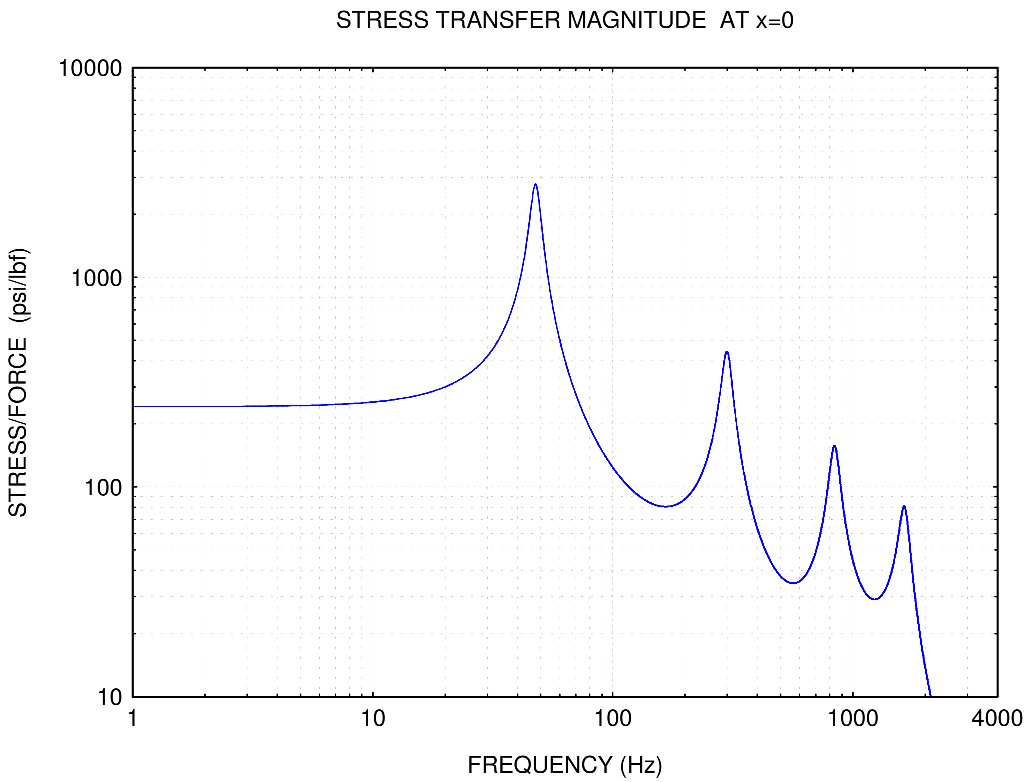

Figure 6.

Reference

1. Weaver, Timoshenko, and Young; Vibration Problems in Engineering, Wiley-Interscience, New York, 1990.

APPENDIX A

Stress-Velocity Relationship

The bending moment is

$$M(x,\omega) = EI \sum_{i=1}^{\infty} \frac{P(\omega)\ Y_i(L)\, Y_i''(x)}{(\omega_i{}^2 - \omega^2)\ + j\ 2\xi_i \omega\ \omega_i} \tag{A-1}$$

The bending stress $\sigma(x,\omega)$ is

$$\sigma(x,\omega) = \frac{\hat{c}}{I} M(x,\omega) \tag{A-2}$$

where

$\hat{c}$ = Distance to neutral axis

$$\sigma(x,\omega) = \hat{c}\, E \sum_{i=1}^{\infty} \frac{P(\omega)\ Y_i(L)\, Y_i''(x)}{(\omega_i{}^2 - \omega^2)\ + j\ 2\xi_i \omega\ \omega_i} \tag{A-3}$$

The bending stress for the first mode at the fixed end is

$$\sigma(0,\omega) = \hat{c}\, E\ \frac{P(\omega)\ Y_1(L)\, Y_i''(0)}{(\omega_1{}^2 - \omega^2)\ + j\ 2\xi_1 \omega\ \omega_1} \tag{A-4}$$

The velocity is

$$\hat{v}(x,\omega) = \sum_{i=1}^{\infty} \frac{j\omega\ P(\omega)\, Y_i(L)\, Y_i(x)}{(\omega_i{}^2 - \omega^2)\ + j\ 2\xi_i \omega\ \omega_i} \tag{A-5}$$

The velocity for the first mode at the free end is

$$\hat{v}(L,\omega) = \frac{j\omega\, P(\omega)\, Y_1(L)\, Y_1(L)}{(\omega_1{}^2 - \omega^2) + j\, 2\xi_1 \omega\, \omega_1} \tag{A-6}$$

$$\sigma(0,\omega) = \hat{c}\, E \; \frac{P(\omega)\; Y_1(L)\, Y_i''(0)}{(\omega_1{}^2 - \omega^2) + j\, 2\xi_1 \omega\, \omega_1} \tag{A-7}$$

The second derivative of the mode shape is

$$Y_i''(x) = \left\{ \frac{\beta_i{}^2}{\sqrt{mL}} \right\} \left\{ \left[\cosh(\beta_i x) + \cos(\beta_i x) \right] - D_i \left[\sinh(\beta_i x) + \sin(\beta_i x) \right] \right\} \tag{A-8}$$

$$Y_i''(0) = \frac{2\beta_i{}^2}{\sqrt{mL}} \tag{A-9}$$

$$\sigma(0,\omega) = \hat{c}\, E\, \frac{2\beta_1{}^2}{\sqrt{mL}} \frac{P(\omega)\; Y_1(L)}{(\omega_1{}^2 - \omega^2) + j\, 2\xi_1 \omega\, \omega_1} \tag{A-10}$$

$$\omega_n = \beta_n{}^2 \sqrt{\frac{EI}{m}} \tag{A-11}$$

$$\beta_n{}^2 = \omega_n \sqrt{\frac{m}{EI}} \tag{A-12}$$

$$\sigma(0,\omega) = \hat{c}\, E \, \frac{2}{\sqrt{mL}} \, \omega_1 \sqrt{\frac{m}{EI}} \, \frac{P(\omega)\; Y_1(L)}{(\omega_1{}^2 - \omega^2) \; + j\; 2\xi_1 \omega\, \omega_1} \tag{A-13}$$

$$\sigma(0,\omega) = 2\,\hat{c} \;\; \omega_1 \sqrt{\frac{E}{IL}} \, \frac{P(\omega)\; Y_1(L)}{(\omega_1{}^2 - \omega^2) \; + j\; 2\xi_1 \omega\, \omega_1} \tag{A-14}$$

Again,

$$\hat{v}(L,\omega) = \frac{j\omega\; P(\omega)\, Y_1(L)\, Y_1(L)}{(\omega_1{}^2 - \omega^2) \; + j\; 2\xi_1 \omega\, \omega_1} \tag{A-15}$$

The bending stress for the first mode at the fixed end is

$$\sigma(0,\omega) = 2\,\hat{c} \;\; \frac{\omega_1}{j\omega} \sqrt{\frac{E}{IL}} \;\; \frac{\hat{v}(L,\omega)}{Y_1(L)} \tag{A-16}$$

The magnitude is

$$\left|\, \sigma(0,\omega) \right| = 2\,\hat{c} \;\; \frac{\omega_1}{\omega} \sqrt{\frac{E}{IL}} \;\; \frac{\left|\, \hat{v}(L,\omega) \right|}{Y_1(L)} \tag{A-17}$$