THE RESPONSE OF A CANTILEVER BEAM SUBJECTED TO AN APPLIED CONCENTRATED HALF-SINE FORCE PULSE
Revision B

By Tom Irvine
Email: tom@vibrationdata.com

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Consider a cantilever beam with an applied force at the free end.

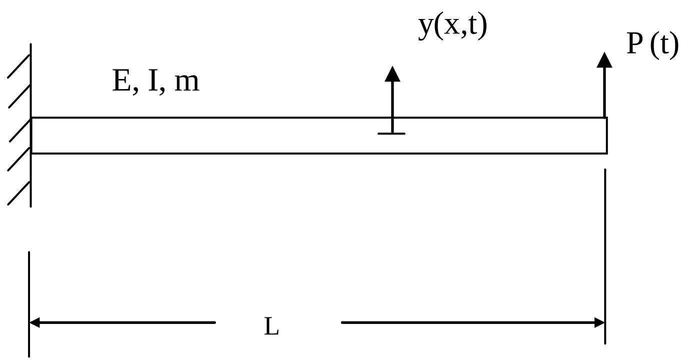


Figure 1.

The variables are

Area moment of inertia	I
Cross-section area	A
Elastic Modulus	E
Length	L
Mass per Volume	ρ
Mass per Length	m
Displacement	u(x,t)
Applied Force	F(t)
Base Excitation Frequency (rad/sec)	ω
Natural Frequency (rad/sec)	ω_n
Viscous Damping Ratio	ξ

Assume uniform mass density and constant cross-section. The governing equation from Reference 1 is

$$-EI\frac{\partial^4 y}{\partial x^4} = m\frac{\partial^2 y}{\partial t^2} \tag{1}$$

The boundary conditions at the fixed end x = 0 for the case without the force are

$$y(0, t) = 0 \qquad \text{(zero displacement)} \tag{2}$$

$$\left.\frac{\partial}{\partial x} y(x,t)\right|_{x=0} = 0 \qquad \text{(zero slope)} \tag{3}$$

The boundary conditions at the free end x = L for the case without the force are

$$\left.\frac{\partial^2}{\partial x^2} y(x,t)\right|_{x=L} = 0 \qquad \text{(zero bending moment)} \tag{4}$$

$$\left.\frac{\partial^3}{\partial x^3} y(x,t)\right|_{x=L} = 0 \qquad \text{(zero shear force)} \tag{5}$$

$$y(x,t) = \sum_{i=1}^{\infty} \phi_i(t)\, Y_i(x) \tag{6}$$

The ϕ_i terms are some unknown functions of time which will be determined by the principle of virtual work.

The natural frequencies are determined from the roots as follows.

Table 1. Roots	
Index	$\beta_n L$
n = 1	1.87510
n = 2	4.69409
n = 3	$5\pi/2$
n = 4	$7\pi/2$

The natural frequency term ω_n is thus

$$\omega_n = \beta_n{}^2 \sqrt{\frac{EI}{m}} \tag{7}$$

The calculation steps are omitted for brevity. The resulting normalized eigenvectors are

$$Y_i(x) = \left\{\frac{1}{\sqrt{mL}}\right\}\left\{\left[\cosh(\beta_i x) - \cos(\beta_i x)\right] - D_i \left[\sinh(\beta_i x) - \sin(\beta_i x)\right]\right\} \tag{8}$$

$$D_i = \frac{\cos(\beta_i L) + \cosh(\beta_i L)}{\sin(\beta_i L) + \sinh(\beta_i L)} \tag{9}$$

The virtual transverse displacement δy_i in terms of the mode shapes are

$$\delta y_i = \left\{\frac{1}{\sqrt{mL}}\right\}\left\{\left[\cosh(\beta_i x) - \cos(\beta_i x)\right] - D_i \left[\sinh(\beta_i x) - \sin(\beta_i x)\right]\right\} \tag{10}$$

The mass of an element between two adjacent cross sections of the rod is $m\,dx$.

The work δW_I done by inertial forces on the assumed virtual displacement is

$$\delta W_I = \int_0^L (-m dx)\, \ddot{y}\, \delta y_i \tag{11}$$

$$\delta y_i = \delta\phi_i\, Y_i \tag{12}$$

$$\delta W_I = -m\, \delta\phi_i \int_0^L \ddot{y}\; Y_i(x)\, dx \tag{13}$$

By substitution,

$$\delta W_I = -m\, \delta\phi_i \int_0^L \left\{ \sum_{j=1}^{\infty} \ddot{\phi}_j(t)\, Y_j(x) \right\} Y_i(x)\, dx \tag{14}$$

$$\delta W_I = -m\; \delta\phi_i \sum_{j=1}^{\infty} \ddot{\phi}_j \int_0^L Y_i(x) Y_j(x)\, dx \tag{15}$$

The orthogonality of the normal mode shapes is such that

$$\int_0^L Y_i(x) Y_j(x)\; dx \; = 0\,, \quad \text{for } i \neq j \tag{16}$$

$$m \int_0^L Y_i(x) Y_j(x)\; dx \; = 1\,, \quad \text{for } i = j \tag{17}$$

$$\delta W_{I\,i} = -\ddot{\phi}_i\, \delta\phi_i \tag{18}$$

Now calculate the strain energy U produced by the elastic forces.

$$U = \int_0^L \frac{EI}{2} \left(y''\right)^2 dx \tag{19}$$

$$U = \frac{EI}{2}\int_0^L \left(\sum_{i=1}^{\infty} \phi_i \, Y_i''\right)\left(\sum_{j=1}^{\infty} \phi_j \, Y_j''\right) dx \tag{20}$$

$$U = \frac{EI}{2}\int_0^L \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \phi_i \, \phi_j Y_i'' \, Y_j'' \; dx \tag{21}$$

$$U = \frac{EI}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \phi_i \, \phi_j \int_0^L Y_i'' \, Y_j'' dx \tag{22}$$

The orthogonality relationships are

$$\int_0^L Y_i'' \, Y_j'' dx = 0, \quad \text{for } i \neq j \tag{23}$$

$$\int_0^L Y_i'' \, Y_j'' dx = \beta_i^{\,4}, \quad \text{for } i = j \tag{24}$$

Thus,

$$U = \frac{EI}{2}\sum_{i=1}^{\infty} \beta_i^{\,4} \, \phi_i^{\,2} \tag{25}$$

The virtual work of the elastic forces is

$$\delta W_{Ei} = -\frac{\partial U}{\partial \phi_i} \delta\phi_i \tag{26}$$

$$\delta W_{Ei} = -\frac{\partial}{\partial \phi_i}\left\{\frac{EI}{2}\sum_{i=1}^{\infty} \beta_i^{4}\, \phi_i^{2}\right\}\delta\phi_i \tag{27}$$

$$\delta W_{Ei} = -EI\,\beta_i^{4}\,\phi_i\,\delta\phi_i \tag{28}$$

Determine the work of the applied concentrated force.

$$\delta W_{Fi} = P(t)\ \delta y_i(L,t) \tag{29}$$

$$\delta W_{Fi} = P(t)\ Y_i(L)\,\delta\phi_i \tag{30}$$

The total virtual work is thus

$$\ddot{\phi}_i\,\delta\phi_i + EI\,\beta_i^{4}\,\phi_i\,\delta\phi_i = P(t)\ Y_i(L)\,\delta\phi_i \tag{31}$$

$$\ddot{\phi}_i + EI\,\beta_i^{4}\,\phi_i = P(t)\ Y_i(L) \tag{32}$$

$$\ddot{\phi}_i + \omega_i^{2}\,\phi_i = P(t)\ Y_i(L) \tag{33}$$

Add modal damping

$$\ddot{\phi}_i + 2\xi_i\omega_i\,\dot{\phi}_i + \omega_i^{2}\,\phi_i = P(t)\ Y_i(L) \tag{34}$$

Time Domain

Let

$$P(t) = \hat{P} \sin(\alpha t) \qquad \text{for } t \leq T \tag{35}$$

where

$$\alpha = \pi / T$$

The modal equation of motion is

$$\ddot{\phi}_i + 2\xi_i \omega_i \dot{\phi}_i + \omega_i^2 \phi_i = P(t) Y_i(L) \tag{36}$$

$$\ddot{\phi}_i + 2\xi_i \omega_i \dot{\phi}_i + \omega_i^2 \phi_i = \hat{P} Y_i(L) \sin(\beta t) \tag{37}$$

Let

$$F_i = \hat{P}\, Y_i(L) \tag{38}$$

$$\ddot{\phi}_i + 2\xi_i \omega_i \dot{\phi}_i + \omega_i^2 \phi_i = F_i \sin(\beta t) \tag{39}$$

Let

$$\omega_{d,i} = \omega_i \sqrt{1 - \xi_i^2} \tag{40}$$

Assume that the initial conditions are zero.

The modal displacement during the half-sine pulse is

$$\phi_i(t) =$$

$$+\frac{F_i}{\left(\alpha^2-\omega_i^2\right)^2+\left(2\xi_i\,\omega\omega_i\right)^2}\left\{-2\xi_i\omega_i\alpha\cos(\alpha t)-\left(\alpha^2-\omega_i^2\right)\sin(\alpha t)\right\}$$

$$+\frac{1}{\omega_{d,i}}\left\{\frac{\alpha\,F_i}{\left(\alpha^2-\omega_i^2\right)^2+\left(2\xi_i\,\alpha\,\omega_i\right)^2}\right\}\left\{e^{-\xi_i\omega_i t}\right\}\left\{2\xi_i\omega_i\omega_{d,i}\cos(\omega_{d,i}t)\right\}$$

$$+\frac{1}{\omega_{d,i}}\left\{\frac{\alpha\,F_i}{\left(\alpha^2-\omega_i^2\right)^2+\left(2\xi_i\alpha\,\omega_i\right)^2}\right\}\left\{e^{-\xi_i\omega_i t}\right\}\left\{\left[\alpha^2+\omega_i^2\left[-1+2\xi_i^2\right]\right]\sin(\omega_{d,i}t)\right\}$$

for $t \le T$

(41)

The modal displacement after the half-sine pulse is

$$\phi_i(\tau)=\exp\left(-\xi_i\omega_i\tau\right)\left\{\phi_i(T)\cos\left(\omega_{d,i}\tau\right)+\left\{\frac{\dot{\phi}_i(T)+\left(\xi_i\,\omega_i\right)\phi_i(T)}{\omega_{d,i}}\right\}\sin\left(\omega_{d,i}\tau\right)\right\},$$

$$\tau = t - T$$

for $t > T$

(42)

The displacement can then be found via

$$y(x,t)=\sum_{i=1}^{\infty}\phi_i(t)\,Y_i(x) \tag{43}$$

Example

Consider the transverse response of an aluminum, fixed-free, circular rod with the following properties.

Length	L	=	24 inch
Diameter	D	=	1 inch
Area	A	=	0.785 inch^2
Area Moment of Inertia	I	=	0.0491 inch^4
Elastic Modulus	E	=	1.0e+07 lbf/in^2
Mass Density	ρ	=	0.1 lbm/in^3
Speed of Sound in Material	c	=	1.96e+05 in/sec
Viscous Damping Ratio	ξ	=	0.05

Four modes are included in the analysis. The natural frequencies are

Table 2. Natural Frequencies

i	f_i (Hz)
1	47.8
2	299
3	837
4	1641

The resulting transfer functions are shown in Figures 2 through 4.

Now apply a 20 lbf, 4 msec half-sine pulse at the free end of the beam.

The following response curves are calculated via Matlab scripts:

cant_beam_half_sine_force.m

frf_from_th.m

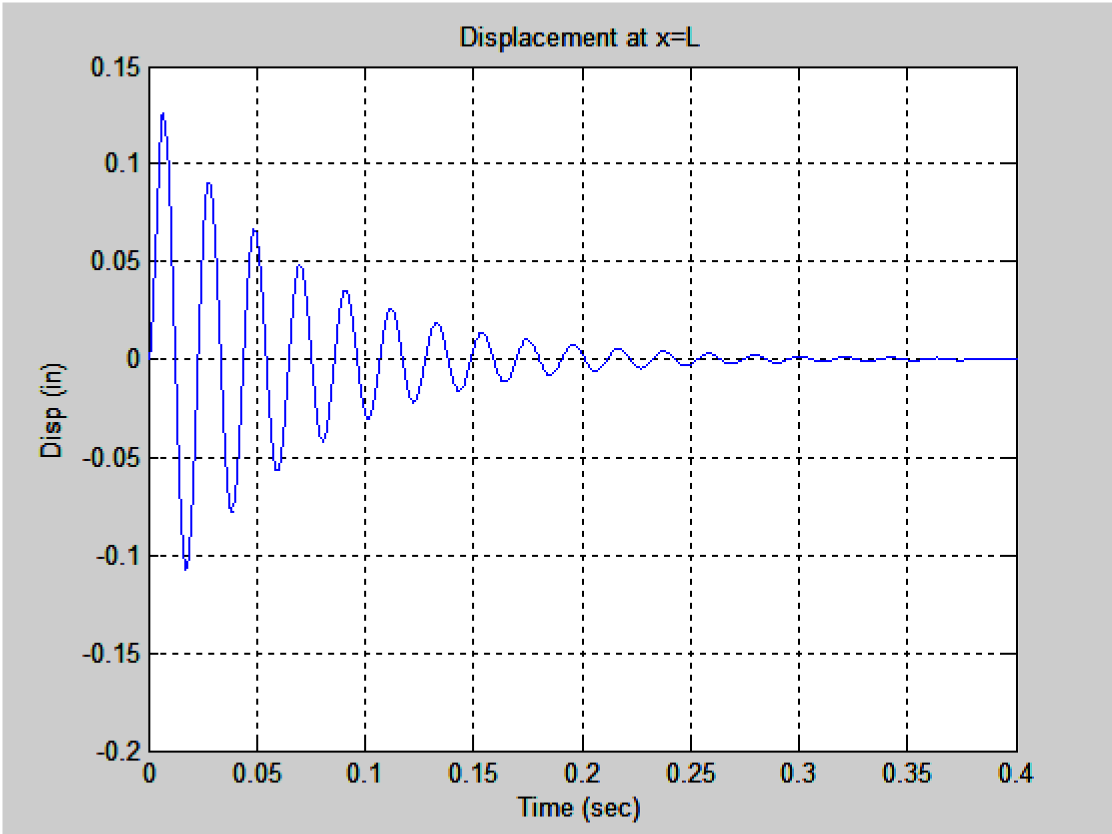


Figure 2.

Table 3. Peak Response Values at x=L	
Parameter	Value
Displacement	0.1257 in
Velocity	34.4 in/sec
Acceleration	55.37 G

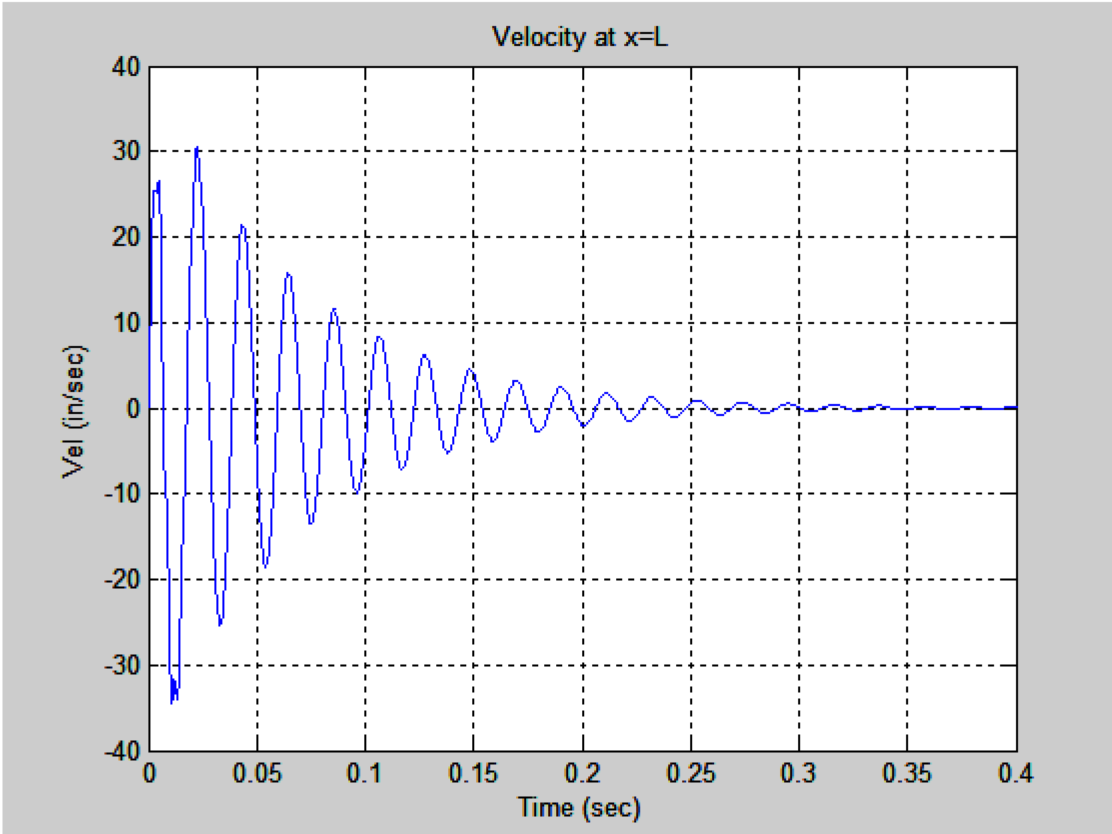

Velocity at x=L
Vel (in/sec)
Time (sec)
40
30
20
10
0
-10
-20
-30
-40
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4

Figure 3.

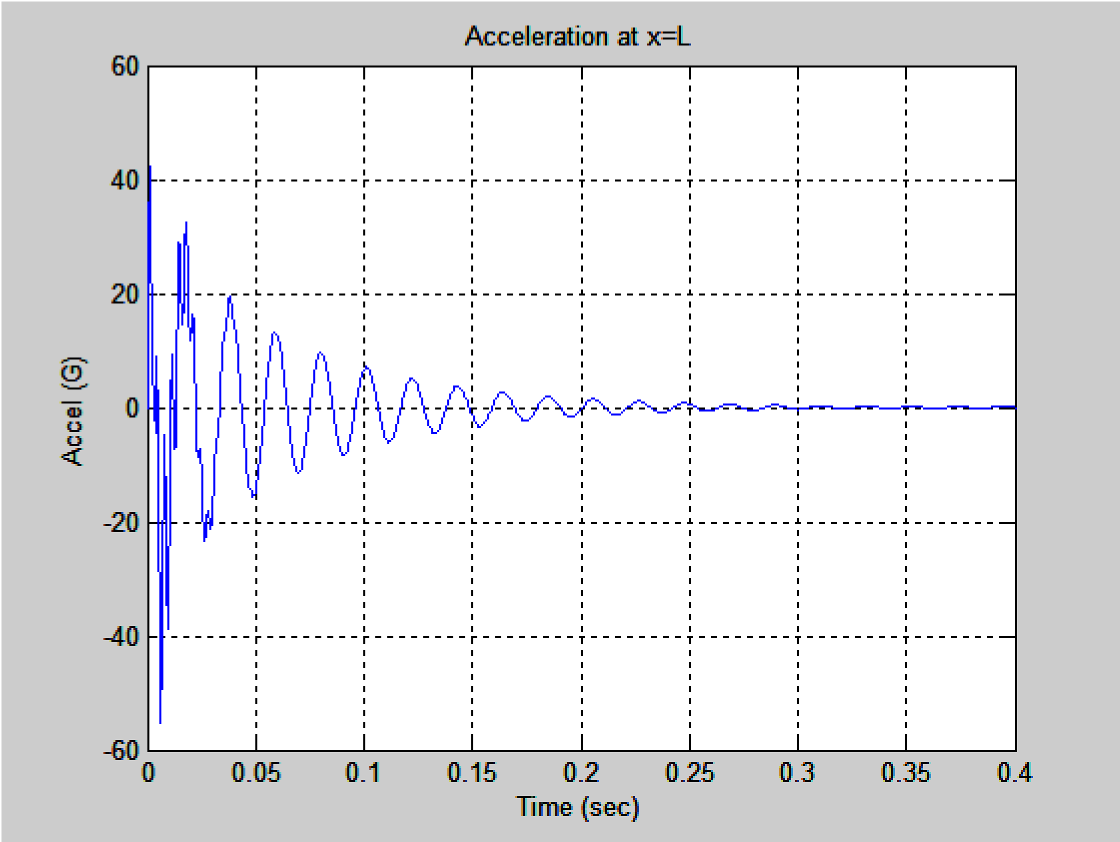

Acceleration at x=L
Accel (G)
60
40
20
0
-20
-40
-60
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
Time (sec)

Figure 4.

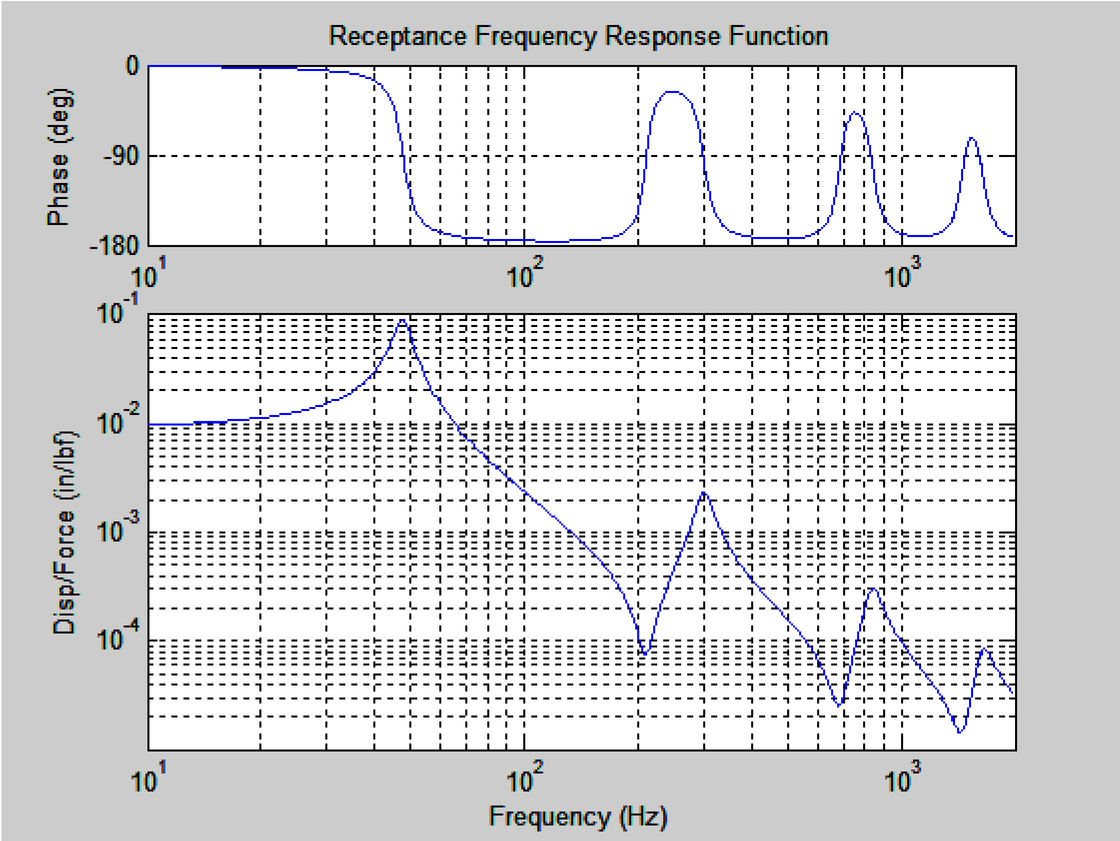
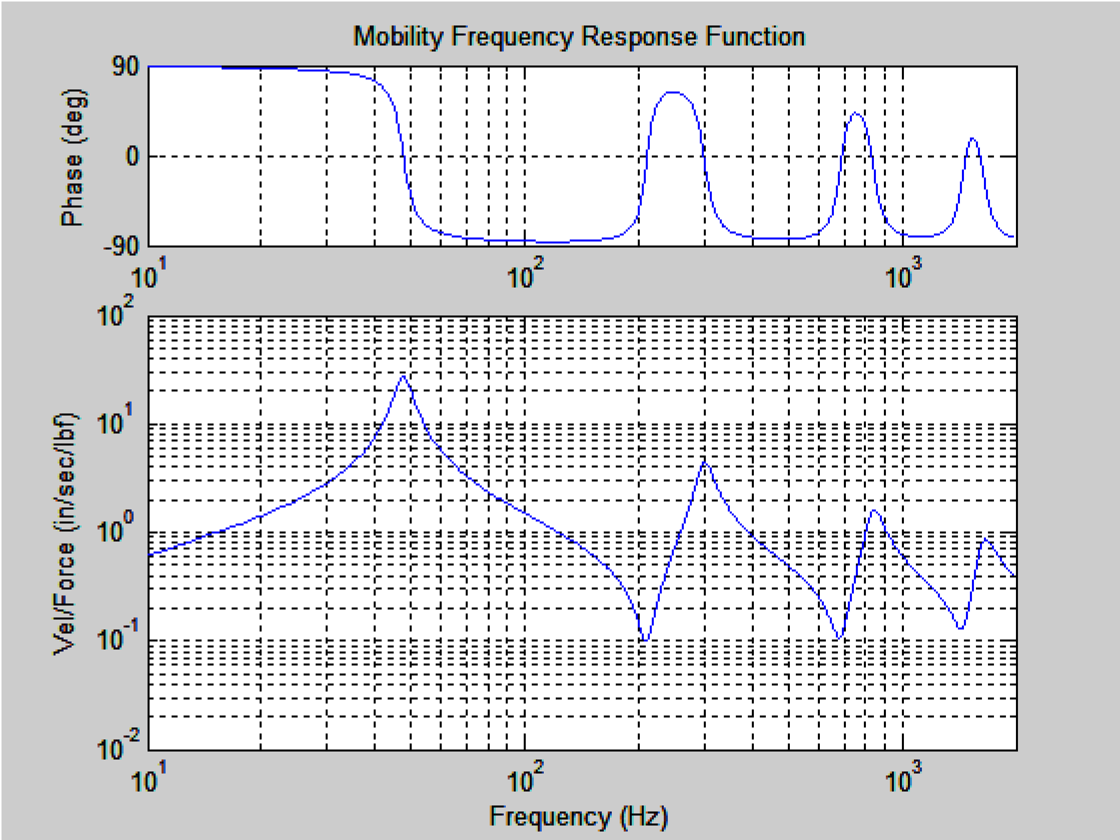

Receptance Frequency Response Function
Phase (deg)
0
-90
-180
10^1
10^2
10^3
Disp/Force (in/lbf)
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^1
10^2
10^3
Frequency (Hz)

Figure 5.



Mobility Frequency Response Function
Phase (deg)
90
0
-90
10^1
10^2
10^3
Vel/Force (in/sec/lbf)
10^2
10^1
10^0
10^{-1}
10^{-2}
10^1
10^2
10^3
Frequency (Hz)

Figure 6.

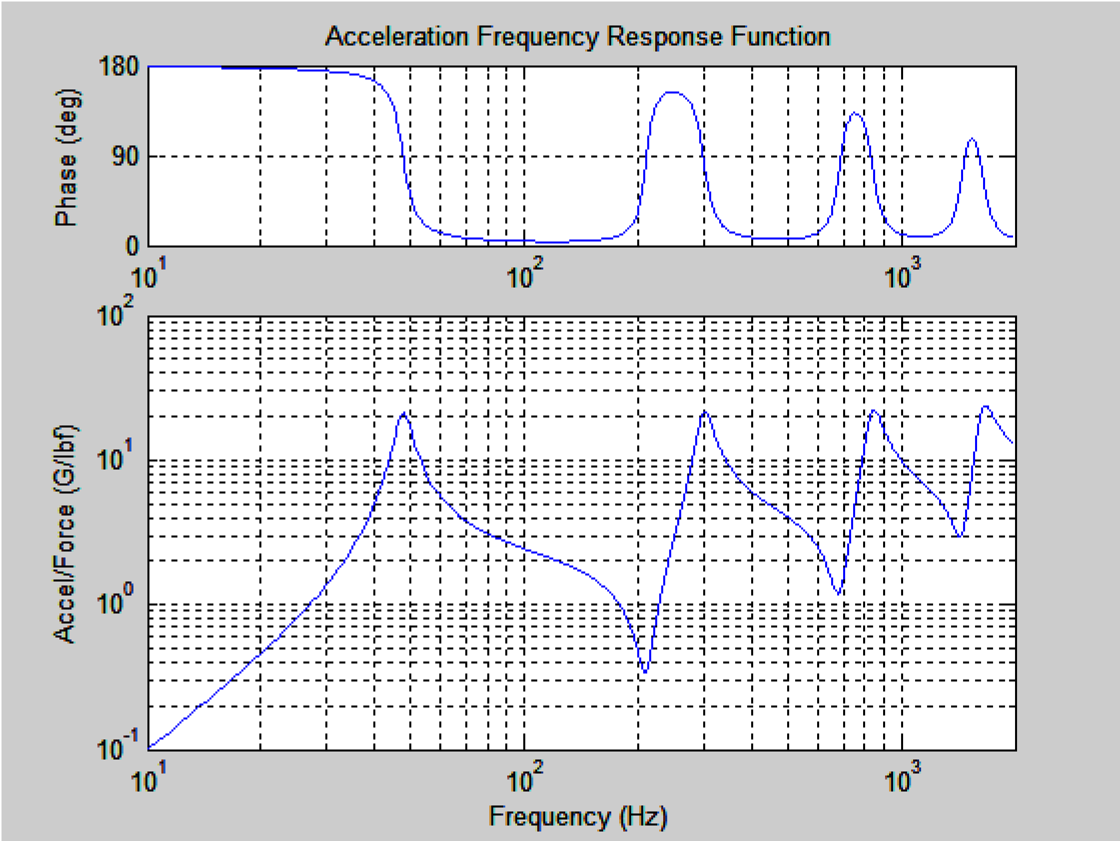


Figure 7.

Reference

1. Weaver, Timoshenko, and Young; Vibration Problems in Engineering, Wiley-Interscience, New York, 1990.