THE RESPONSE OF A CANTILEVER BEAM SUBJECTED TO AN APPLIED CONCENTRATED IMPULSE FORCE

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Consider a cantilever beam with an applied force at the free end.

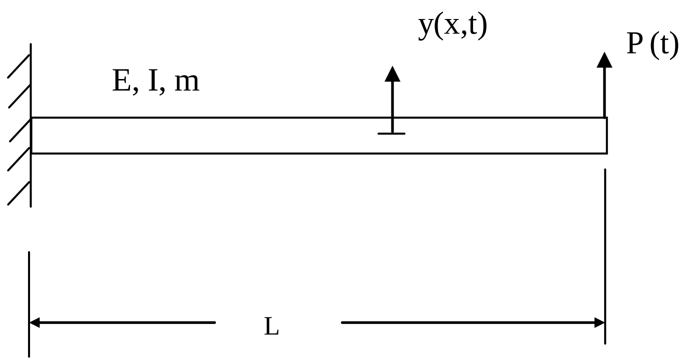

Figure 1.

The variables are

Area moment of inertia	I
Cross-section area	A
Elastic Modulus	E
Length	L
Mass per Volume	ρ
Mass per Length	m
Displacement	u(x,t)
Applied Force	F(t)
Base Excitation Frequency (rad/sec)	ω
Natural Frequency (rad/sec)	ω_n
Viscous Damping Ratio	ξ

Assume uniform mass density and constant cross-section. The governing equation from Reference 1 is

$$-EI\frac{\partial^4 y}{\partial x^4} = m\frac{\partial^2 y}{\partial t^2} \tag{1}$$

The boundary conditions at the fixed end x = 0 for the case without the force are

$$y(0, t) = 0 \qquad \text{(zero displacement)} \tag{2}$$

$$\left.\frac{\partial}{\partial x} y(x,t)\right|_{x=0} = 0 \qquad \text{(zero slope)} \tag{3}$$

The boundary conditions at the free end x = L for the case without the force are

$$\left.\frac{\partial^2}{\partial x^2} y(x,t)\right|_{x=L} = 0 \qquad \text{(zero bending moment)} \tag{4}$$

$$\left.\frac{\partial^3}{\partial x^3} y(x,t)\right|_{x=L} = 0 \qquad \text{(zero shear force)} \tag{5}$$

$$y(x,t) = \sum_{i=1}^{\infty} \phi_i(t)\, Y_i(x) \tag{6}$$

The ϕ_i terms are some unknown functions of time which will be determined by the principle of virtual work.

The natural frequencies are determined from the roots as follows.

Table 1. Roots	
Index	$\beta_n L$
n = 1	1.87510
n = 2	4.69409
n = 3	$5\pi/2$
n = 4	$7\pi/2$

The natural frequency term ω_n is thus

$$\omega_n = \beta_n{}^2 \sqrt{\frac{EI}{m}} \tag{7}$$

The calculation steps are omitted for brevity. The resulting normalized eigenvectors are

$$Y_i(x) = \left\{\frac{1}{\sqrt{mL}}\right\}\left\{\left[\cosh(\beta_i x) - \cos(\beta_i x)\right] - D_i \left[\sinh(\beta_i x) - \sin(\beta_i x)\right]\right\} \tag{8}$$

$$D_i = \frac{\cos(\beta_i L) + \cosh(\beta_i L)}{\sin(\beta_i L) + \sinh(\beta_i L)} \tag{9}$$

The virtual transverse displacement δy_i in terms of the mode shapes are

$$\delta y_i = \left\{\frac{1}{\sqrt{mL}}\right\}\left\{\left[\cosh(\beta_i x) - \cos(\beta_i x)\right] - D_i \left[\sinh(\beta_i x) - \sin(\beta_i x)\right]\right\} \tag{10}$$

The mass of an element between two adjacent cross sections of the rod is $m\,dx$.

The work δW_I done by inertial forces on the assumed virtual displacement is

$$\delta W_I = \int_0^L \left(-\,\mathrm{mdx}\right)\ddot{y}\,\delta y_i \tag{11}$$

$$\delta y_i = \delta\phi_i\, Y_i \tag{12}$$

$$\delta W_I = -m\,\delta\phi_i \int_0^L \ddot{y}\; Y_i(x)\,dx \tag{13}$$

By substitution,

$$\delta W_I = -m\,\delta\phi_i \int_0^L \left\{ \sum_{j=1}^{\infty} \ddot{\phi}_j(t)\, Y_j(x) \right\} Y_i(x)\,dx \tag{14}$$

$$\delta W_I = -m\;\delta\phi_i \sum_{j=1}^{\infty} \ddot{\phi}_j \int_0^L Y_i(x) Y_j(x)\,dx \tag{15}$$

The orthogonality of the normal mode shapes is such that

$$\int_0^L Y_i(x) Y_j(x)\;dx \;= 0\,, \quad \text{for } i \neq j \tag{16}$$

$$m\int_0^L Y_i(x) Y_j(x)\;dx \;= 1\,, \quad \text{for } i = j \tag{17}$$

$$\delta W_{Ii} = -\ddot{\phi}_i\;\delta\phi_i \tag{18}$$

Now calculate the strain energy U produced by the elastic forces.

$$U = \int_0^L \frac{EI}{2}\left(y''\right)^2 dx \tag{19}$$

$$U = \frac{EI}{2}\int_0^L \left(\sum_{i=1}^{\infty} \phi_i \, Y_i''\right)\left(\sum_{j=1}^{\infty} \phi_j \, Y_j''\right) dx \tag{20}$$

$$U = \frac{EI}{2}\int_0^L \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \phi_i \, \phi_j Y_i'' \, Y_j'' \; dx \tag{21}$$

$$U = \frac{EI}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \phi_i \, \phi_j \int_0^L Y_i'' \, Y_j'' dx \tag{22}$$

The orthogonality relationships are

$$\int_0^L Y_i'' \, Y_j'' dx = 0, \quad \text{for } i \neq j \tag{23}$$

$$\int_0^L Y_i'' \, Y_j'' dx = \beta_i^4, \quad \text{for } i = j \tag{24}$$

Thus,

$$U = \frac{EI}{2}\sum_{i=1}^{\infty} \beta_i^4 \, \phi_i^2 \tag{25}$$

The virtual work of the elastic forces is

$$\delta W_{Ei} = -\frac{\partial U}{\partial \phi_i} \delta\phi_i \tag{26}$$

$$\delta W_{Ei} = -\frac{\partial}{\partial \phi_i}\left\{\frac{EI}{2}\sum_{i=1}^{\infty}\beta_i^4 \phi_i^2\right\}\delta\phi_i \tag{27}$$

$$\delta W_{Ei} = -EI\,\beta_i^4\,\phi_i\,\delta\phi_i \tag{28}$$

Determine the work of the applied concentrated force.

$$\delta W_{Fi} = P(t)\;\delta y_i(L,t) \tag{29}$$

$$\delta W_{Fi} = P(t)\;Y_i(L)\,\delta\phi_i \tag{30}$$

The total virtual work is thus

$$\ddot{\phi}_i\,\delta\phi_i + EI\,\beta_i^4\,\phi_i\,\delta\phi_i = P(t)\;Y_i(L)\,\delta\phi_i \tag{31}$$

$$\ddot{\phi}_i + EI\,\beta_i^4\,\phi_i = P(t)\;Y_i(L) \tag{32}$$

$$\ddot{\phi}_i + \omega_i^2\,\phi_i = P(t)\;Y_i(L) \tag{33}$$

Add modal damping

$$\ddot{\phi}_i + 2\xi_i\omega_i\,\dot{\phi}_i + \omega_i^2\,\phi_i = P(t)\;Y_i(L) \tag{34}$$

<u>Time Domain</u>

The following approach is taken from Reference 2.

Let $\hat{I}$ = total impulse.

$$P(t) = \hat{I}\ \delta(t) \tag{35}$$

$$\ddot{\phi}_i + 2\xi_i\omega_i\dot{\phi}_i + \omega_i^2\phi_i = \hat{I}\ \delta(t) Y_i(L) \tag{36}$$

Let

$$\omega_{d,i} = \omega_i\sqrt{1-\xi_i^{\ 2}} \tag{37}$$

Assume that the initial conditions are zero.

The modal displacement is

$$\phi_i(t) = \hat{I}\left[\frac{1}{\omega_{d,i}}\right] Y_i(L) \exp(-\xi_i\omega_i t)\sin(\omega_{d,i} t) \tag{38}$$

The modal velocity is

$$\dot{\phi}_i(t) = \hat{I}\left[\frac{1}{\omega_{d,i}}\right] Y_i(L)\left\{-\xi_i\omega_i \sin(\omega_{d,i} t) + \omega_{d,i}\cos(\omega_{d,i} t)\right\}\exp(-\xi_i\omega_i t) \tag{39}$$

$$\dot{\phi}_i(t) = -\xi_i\omega_i\phi_i + \hat{I}\ Y_i(L)\left\{\cos(\omega_{d,i} t)\right\}\exp(-\xi_i\omega_i t) \tag{40}$$

The modal acceleration is

$$\begin{aligned}\ddot{\phi}_i &= -\xi_i\omega_i\dot{\phi}_i \\ &\quad + \hat{I}\ Y_i(L)\left\{-\xi_i\omega_i\cos(\omega_{d,i} t) - \omega_{d,i}\sin(\omega_{d,i} t)\right\}\exp(-\xi_i\omega_i t)\end{aligned} \tag{41}$$

$$\ddot{\phi}_i = -\xi_i \omega_i \dot{\phi}_i + \hat{I}\left[\frac{1}{\omega_{d,i}}\right] Y_i(L)\left\{-\xi_i \omega_i \omega_{d,i} \cos(\omega_{d,i} t) - \omega_i^2 \left(1-\xi_i^2\right) \sin(\omega_{d,i} t)\right\} \exp(-\xi_i \omega_i t) \tag{42}$$

$$\ddot{\phi}_i = -\xi_i \omega_i \dot{\phi}_i - \omega_i^2 \phi_i + \hat{I}\left[\frac{1}{\omega_{d,i}}\right] Y_i(L)\left\{-\xi_i \omega_i \omega_{d,i} \cos(\omega_{d,i} t) + \xi_i^2 \omega_{n,i}^2 \sin(\omega_{d,i} t)\right\} \exp(-\xi_i \omega_i t) \tag{43}$$

$$\ddot{\phi}_i = -\xi_i \omega_i \dot{\phi}_i - \omega_i^2 \phi_i + \hat{I}\left[\frac{\xi_i \omega_i}{\omega_{d,i}}\right] Y_i(L)\left\{-\omega_{d,i} \cos(\omega_{d,i} t) + \xi_i \omega_{n,i} \sin(\omega_{d,i} t)\right\} \exp(-\xi_i \omega_i t) \tag{44}$$

$$\ddot{\phi}_i = -2\xi_i \omega_i \dot{\phi}_i - \omega_i^2 \phi_i \tag{45}$$

The physical displacement can then be found via

$$y(x,t) = \sum_{i=1}^{\infty} \phi_i(t)\, Y_i(x) \tag{46}$$

A challenge is to determine the number of modes to include in the analysis. Most of the displacement occurs due to the response of the fundamental mode. The peak velocity and peak acceleration values, however, continue to rise as additional modes are included.

Further consideration is needed.

References

1. Weaver, Timoshenko, and Young; Vibration Problems in Engineering, Wiley-Interscience, New York, 1990.
2. T. Irvine, The Time-Domain Response of a Single-degree-of-freedom System Subjected to an Impulse Force, Revision B, Vibrationdata, 2012.