FLUTTER OF A CANTILEVER WING IN A STEADY AIRFLOW

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ABSTRACT

A stability analysis of combined bending and torsional vibrations of a cantilever wing in steady airflow is presented. The system is non-self-adjoint and consists of two linear homogeneous partial differential equations with space-dependent coefficients. The system admits no closed-form solution, so that an approximate method is used. In particular, Galerkin's method is used in conjunction with finite series of comparison functions for the bending and torsional displacements. The method results in a discretized eigenvalue problem in the form of an algebraic eigenvalue problem.

A numerical example reveals that for a given set of system parameters the third mode is responsible for the first divergent flutter condition. Consequently, in this case the bending and torsional displacements must be described by at least two comparison functions each to determine the flutter condition.

Introduction

The combined bending and torsional vibrations of a cantilever wing in a steady airflow is treated under the quasi-steady assumption [1]. Free vibration of the wing is being investigated so that the only external

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forces acting on the wing are the aerodynamic forces as shown in Fig. 1. These forces are found using the strip theory approximation in which the lift coefficient is proportional to the local geometric angle of attack [2]. Fung [1] has derived the governing equations of motion and the linear and angular displacements are each approximated by one assumed mode. It is shown in this paper, through an example, that such an approximation may be inadequate, even as a first approximation, depending on the nature of the flutter condition. Therefore a more complete analysis is presented by means of Galerkin's method, in which the bending and torsional displacements are each described by a finite series of comparison functions. Standard application of the method leads to an algebraic eigenvalue problem with the airflow velocity relative to the wing as a parameter.

Equations of Motion

Let the bending deflection of the elastic axis and the rotation of the elastic axis be denoted by $w(x,t)$ and $\theta(x,t)$, respectively, where $w(x,t)$ is positive downward and $\theta(x,t)$ is positive if the leading edge is up, where the latter is known as the local angle of attack. The distance between the leading edge and the elastic axis is denoted by $y_o(t)$ and the distance between the elastic axis and the inertial axis is denoted by $y_\theta(t)$. The chord length is $c(x)$. The velocity of the airflow relative to the wing is a constant and denoted by V. The two partial differential equations of motion have been shown to be [1]

$$\frac{\partial}{\partial x^2}\left(EI\,\frac{\partial^2 w}{\partial x^2}\right) + m\,\frac{\partial^2 w}{\partial t^2} + m y_\theta\,\frac{\partial^2 \theta}{\partial t^2}$$

$$+ \frac{\rho V^2}{2}\,c\left[\theta + \frac{1}{V}\frac{\partial w}{\partial t} + \frac{c}{V}\left(\frac{3}{4} - \frac{y_o}{c}\right)\frac{\partial \theta}{\partial t}\right] = 0 \qquad (1a)$$

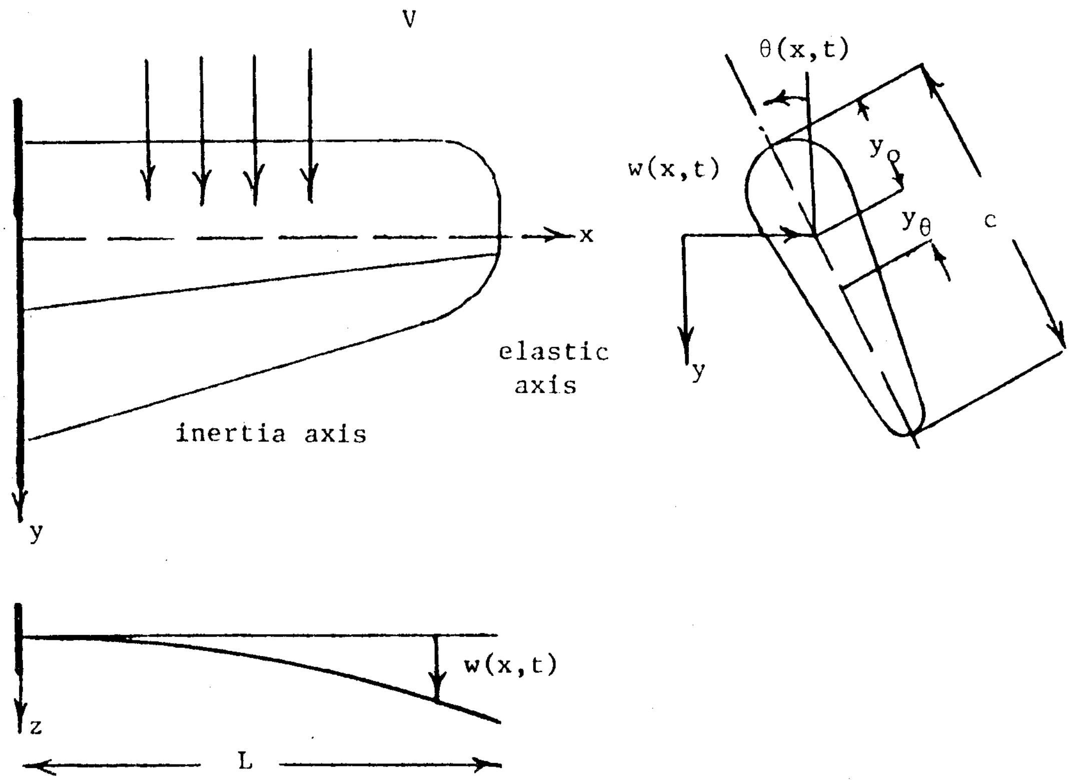


FIGURE 1

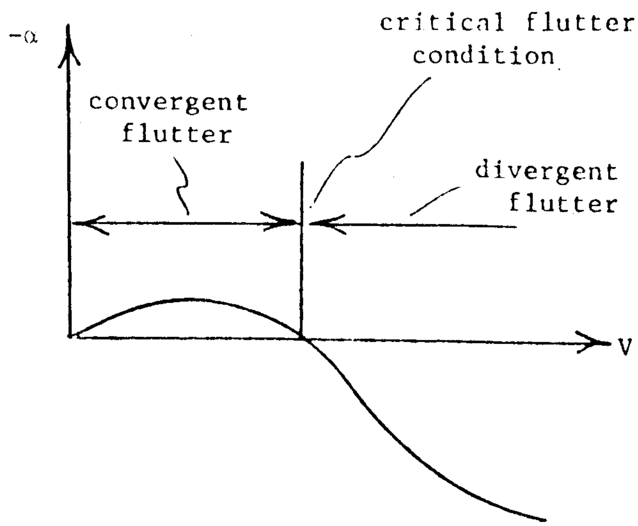


FIGURE 2

$$-\frac{\partial}{\partial x}\,(GJ\,\frac{\partial\theta}{\partial x}) + my_\theta\,\frac{\partial^2 w}{\partial t^2} + I_\theta\,\frac{\partial^2\theta}{\partial t^2} - \frac{\rho V^2}{2}\,c^2\,\{\frac{-c\pi}{8V}\,\frac{\partial\theta}{\partial t}$$

$$+ (\frac{y_o}{c} - \frac{1}{4})\,\frac{dc_L}{d\theta}\,[\theta + \frac{1}{V}\,\frac{\partial w}{\partial t} + \frac{c}{v}\,(\frac{3}{4} - \frac{y_o}{c})\,\frac{\partial\theta}{\partial t}]\} = 0 \qquad (1b)$$

where EI(x) and GJ(x) are the bending and torsional stiffnesses, respectively, m(x) is the mass per unit length, ρ denotes the air density and c_L is the local lift coefficient. The aerodynamic forces and moments were obtained using "strip theory" in which c_L is assumed to be proportional to the local angle of attack and is such that $dc_L/d\theta = 2\pi$. It is further assumed that the airfoil has circular or parabolic camber. Moreover, the analysis is subject to the quasi-steady assumption [1].

Equations (1) are linear and homogeneous with space-dependent coefficients. The displacements w and θ are subjected to the homogeneous boundary conditions

$$w(x,t)\big|_{x=0} = \frac{\partial}{\partial x}\,w(x,t)\big|_{x=0} = \theta(x,t)\big|_{x=0} = 0 \qquad (2a)$$

$$EI\,\frac{\partial^2 w}{\partial x^2}\Big|_{x=L} = \frac{\partial}{\partial x}\,(EI\,\frac{\partial^2 w}{\partial x^2})\Big|_{x=L} = \frac{\partial}{\partial x}\,[GJ\,\frac{\partial\theta}{\partial x}]\Big|_{x=L} = 0 \qquad (2b)$$

It is of immediate concern to formulate the eigenvalue problem. As we are about to see, the system is non-self-adjoint. Assuming a solution in the form

$$w(x,t) = w(x)e^{\lambda t} \quad , \quad \theta(x,t) = \Theta(x)e^{\lambda t} \qquad (3a,b)$$

where λ is in general complex, introducing Eqs. (3) into Eqs. (1) and dividing through by $e^{\lambda t}$ we have

$$(EIW'')'' + \frac{\rho V^2}{2} c \frac{dc_L}{d\theta} \Theta + \lambda \frac{\rho V}{2} c \frac{dc_L}{d\theta} [W + c [\frac{3}{4} - \frac{y_o}{c}]\Theta]$$

$$+ \lambda^2 m[W + y_\theta \Theta] = 0 \qquad (4a)$$

$$-[GJ\Theta']' - \frac{\rho V^2}{2} c^2 (\frac{y_o}{c} - \frac{1}{4}) \frac{dc_L}{d\theta} \Theta - \lambda \frac{\rho V}{2} c^2 \{(\frac{y_o}{c} - \frac{1}{4}) \frac{dc_L}{d\theta} W$$

$$+ c[(\frac{y_o}{c} - \frac{1}{4})(\frac{3}{4} - \frac{y_o}{c}) \frac{dc_L}{d\theta} - \frac{\pi}{8}]\Theta\} + \lambda^2 (my_\theta W + I_\theta \Theta) = 0 \qquad (4b)$$

The boundary conditions retain the form (2), but with $W(x)$ replacing $w(x,t)$ and $\Theta(x)$ replacing $\theta(x,t)$. Partial derivatives become total derivatives with respect to x.

An Approximate Solution

No closed-form solution to Eqs. (4) exists. Therefore an approximate solution is sought by means of Galerkin's method [3]. The solutions are assumed to be in the form

$$W = \sum_{j=1}^{n} a_j \psi_j \quad , \quad \Theta = \sum_{j=n+1}^{n+m} a_j \psi_j \qquad (5a,b)$$

where ψ_j $j = 1,2,\ldots,n+m$ are comparison functions multiplied by the constants a_j $j = 1,2,\ldots,n+m$. The first n functions must be four times differentiable and satisfy the boundary conditions

$$\psi_j|_{x=0} = \psi_j'|_{x=0} = EI\psi_j''|_{x=L} = (EI\psi_j'')'|_{x=L} = 0 \ . \qquad (6a)$$

The remaining m functions must be twice differentiable and satisfy

$$\psi_j|_{x=L} = GJ\psi_j'|_{x=L} = 0 \ . \qquad (6b)$$

Substitution of Eqs. (5) into Eqs. (4) yields

$$\sum_{j=1}^{n} a_j \ (EI\psi_j'')'' + \lambda^2 \sum_{j=1}^{n} a_j, \psi_j + \lambda^2 \sum_{j=n+1}^{n+m} a_j m y_\theta \psi_j$$

$$+ V^2 \sum_{j=n+1}^{n+m} \frac{\rho}{2} c^2 \frac{dc_L}{d} a_j \psi_j + \lambda V \sum_{j=1}^{n} \frac{\rho}{2} c \frac{dc_L}{d\theta} a_j \psi_j$$

$$+ \lambda V \sum_{j=n+1}^{n+m} \frac{\rho}{2} c^2 \frac{dc_L}{d\theta} (\frac{3}{4} - \frac{y_o}{c}) a_j \psi_j = 0 \qquad (7a)$$

$$- \sum_{j=n+1}^{n+m} a_j (GJ\psi_j')' + \lambda^2 \sum_{j=1}^{n} a_j m y_\theta \psi_j + \theta^2 \sum_{j=n+1}^{n+m} a_j I_\theta \psi_j$$

$$- V^2 \sum_{j=n+1}^{n+m} \frac{\rho}{2} c^2 (\frac{y_o}{c} - \frac{1}{4}) \frac{dc_L}{d\theta} a_j \psi_j - \lambda V \sum_{j=1}^{n} \frac{\rho}{2} c^2 (\frac{y_o}{c} - \frac{1}{4}) \frac{dc_L}{d\theta} a_j \psi_j$$

$$- \lambda V \sum_{j=n+1}^{n+m} \frac{\rho}{2} c^3 \ [(\frac{y_o}{c} - \frac{1}{4})(\frac{3}{4} - \frac{y_o}{c}) \frac{dc_L}{d\theta} - \frac{\pi}{8}) \ a_j \psi_j = 0 \qquad (7b)$$

Next multiply Eq. (7a) by ψ_i $i=1,2,3,\ldots,n$ and Eq. (7b) by ψ_j $j = 1,2,\ldots,n$, integrate both results over the domain $0 \leq x \leq L$ and obtain the algebraic eigenvalue problem

$$\sum_{i=1}^{n+m} k_{ij} a_j + V^2 \sum_{i=1}^{n+m} h_{ij} a_j + \lambda V \sum_{i=1}^{n+m} \ell_{ij} a_j + \lambda^2 \sum_{i=1}^{n+m} m_{ij} a_j = 0 \quad (8)$$

where

$$k_{ij} = k_{ji} = \int_o^L \rho_i (EI\psi_j'')''dx = \int_o^L EI\psi_i''\psi_j dx$$

$$i=1,2,\ldots,n \quad , \quad j=1,2,\ldots,n \tag{9a}$$

$$k_{ij} = k_{ji} = 0 \quad i=1,2,\ldots,n \quad j=n+1,\ldots,n+m$$

$$k_{ij} = -\int_o^L \psi_i (GJ\psi_j')'dx = \int_o^L GJ\psi_i'\psi_j'dx$$

$$i=n+1,\ldots,n+m \quad , \quad j=n+1,\ldots,n+m$$

$$m_{ij} = m_{ji} = \int_o^L m\psi_i\psi_j dx \quad i=1,2,\ldots,n \quad , \quad j=1,2,\ldots,n$$

$$m_{ij} = m_{ji} = \int_o^L my_\theta\psi_i\psi_j dx \quad i=1,2,\ldots,n \quad , \quad j=n+1,\ldots,n+m \tag{9b}$$

$$m_{ij} = m_{ji} = \int_o^L I_\theta\psi_i\psi_j dx \quad i=n+1,\ldots,n+m \quad , \quad j=n+1,\ldots,n+m$$

$$h_{ij} = 0 \quad i=1,2,\ldots,n \quad j=1,2,\ldots,n$$

$$h_{ij} = \frac{\rho}{2}\frac{dc_L}{d\theta}\int_o^L c\psi_i\psi_j dx \quad i=1,2,\ldots,n \quad j=n+1,\ldots,n+m$$

$$h_{ij} = 0 \quad i=n+1,\ldots,n+m \quad j=1,2,\ldots,n \tag{9c}$$

$$h_{ij} = -\frac{\rho}{2}\frac{dc_L}{d\theta}\int_o^L c^3 \left(\frac{y_o}{c} - \frac{1}{4}\right)\psi_i\psi_j dx$$

$$i=n+1,\ldots,n+m \quad , \quad j=n+1,\ldots,n+m.$$

$$\ell_{ij} = \frac{\rho}{2}\frac{dc_L}{d\theta}\int_o^L c\psi_i\psi_j dx \quad i=1,2,\ldots,n \quad , \quad j=1,2,\ldots,n$$

$$\ell_{ij} = \frac{\rho}{2}\frac{dc_L}{d\theta}\int_o^L c^2\left(\frac{3}{4} - \frac{y_o}{c}\right)\psi_i\psi_j dx \tag{9d}$$

$$i=1,2,\ldots,n \quad , \quad j=n+1,\ldots,n+m$$

$$\ell_{ij} = -\frac{\rho}{2}\frac{dc_L}{d\theta}\int_o^L c^2\left(\frac{y_o}{c}-\frac{1}{4}\right)\psi_i\psi_j dx$$

$$i = n+1,\ldots,n+m \quad j = 1,2,\ldots,n$$

$$\ell_{ij} = \frac{\rho}{2}\int_o^L c^3 \left[\frac{\pi}{8} - \left(\frac{y_o}{c}-\frac{1}{4}\right)\left(\frac{3}{4}-\frac{y_o}{c}\right)\frac{dc_L}{d\theta}\right]\psi_i\psi_j dx$$

$$i = n+1,\ldots,n+m \quad , \quad j = n+1,\ldots,n+m.$$

Equation (8) can be written in the compact matrix form

$$[K + V^2H + \lambda VL + \lambda^2 M]\underset{\sim}{a} = \underset{\sim}{0} \tag{10}$$

Note from Eq. (10) that when no air flows over the wing, $V = 0$, the algebraic eigenvalue problem reduces to the standard form

$$K\underset{\sim}{a} = -\lambda^2 M\underset{\sim}{a} \tag{11}$$

where K and M are symmetric positive definite matrices and $\lambda = \pm i\omega$ where ω is the frequency of vibration. Hence, in the absence of air flow the system is self-adjoint and positive definite, as expected.

The general system described by Eq. (10), however, is non-self-adjoint, so that $\lambda = \alpha \pm i\omega$ is in general complex (see Fig. 2). Because λ is a continuous complex function of the velocity V it follows that the real part α is a continuous function of V. For sufficiently small V, the general system is in a convergent flutter condition and the wing is assymptotically stable. However, beyond some critical velocity, $V = V_{cr}$, corresponding to $\alpha = 0$, we find a divergent flutter condition and the wing becomes unstable. This critical velocity can be determined from the eigenvalue problem, Eq. (10). To this end, Eq. (10) can be re-written in standard form

$$K^* \underset{\sim}{a}^* = \lambda M^* \underset{\sim}{a}^* \tag{12}$$

where $\underset{\sim}{a}^* = [\underset{\sim}{a}^T \lambda \underset{\sim}{a}^T]^T$ is a 2(n+m) vector and

$$K^* = \left[\begin{array}{c|c} 0 & I \\ \hline -(K+V^2H) & -VL \end{array}\right] \tag{13a}$$

$$M^* = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & M \end{array}\right] \tag{13b}$$

are 2(n+m)×2(n+m) matrices, in which I is the identity matrix of order n+m. Solving the eigenvalue problem (12) for increasing values of V, one can determine whether or not the system can experience flutter and at which critical velocity V_{cr}.

Example

A uniform wing is considered. Referring to Eqs. (5), the comparison functions chosen are the modes of deformation of a uniform cantilever beam and of a uniform torsional bar [3].

$$\psi_i(x) = (\sin\beta_i L - \sinh\beta_i L)(\sin\beta_i x - \sinh\beta_i x)$$

$$+ (\cos\beta_i L + \cosh\beta_i L)(\cos\beta_i x - \cosh\beta_i x) \;,\; i=1,2,\ldots,n \tag{14a}$$

$$\psi_{i+n}(x) = \sin(\frac{2i-1}{2L} x) \quad i = 1, 2, \ldots, m \tag{14b}$$

The parameters of the system are given in Table 1. Computations were performed by distinguishing between three cases. In the first case, one bending mode and one torsional mode were taken. Hence, from Eqs. (14) we have n=m=1. In the second case, the first two bending modes and the first two torsional modes were chosen so that n=m=2. In the third case, three bending modes and three torsional modes were taken so that n=m=3. The resulting eigenvalues are tabulated in Tables 2, 3, 4. In Fig. 3 the real part of the first three eigenvalues is plotted as a function of the air velocity for the three cases indicated.

Discussion

Clearly, the accuracy of the computed eigenvalues depends on the order of the discretization model. Indeed, as more comparison functions are chosen, the error in the computed eigenvalues decreases. In fact, in the limit, as the number of functions chosen approaches infinity, the approximate solution approaches the exact solution, provided the chosen set of comparison functions is complete. It is well known that the set of comparison functions used in this example, the bending and torsional modes, is complete. From Table 2, we conclude that the convergence of the computed eigenvalues to their exact values is relatively fast. Consider the conservative system. In the first case, the natural frequency of the second mode is quite inaccurate. In general, the highest frequencies of the model are inaccurate, so that the first case accurately determines the first frequency of the continuous system, the second case describes the first three frequencies relatively well, etc.

Now consider the general system. In Fig. 3 the relationship between the real part of the eigenvalue to the air velocity is shown. In the first case, a critical flutter condition is reached in the second mode. In the second and third cases, the third mode is responsible for the first divergent flutter condition, hence it is concluded that the third mode is responsible for the first divergent flutter condition.

In this example, where uniform properties were chosen, flutter occurred in the third mode. Therefore an approximation that only characterizes the first two modes of vibration will be unable to describe the flutter condition regardless of the comparison functions chosen. In fact, for the class of non-self-adjoint systems the convergence of the eigenvalues is not governed by an inclusion principle [4] hence trucation of the second and higher modes, even as a first approximation, should be avoided unless specifically justifiable.

References

1. Fung, Y.C., An Introduction to the Theory of Aeroelasticity, Dover Publications, New York, 1969.

2. Milne-Thompson, L.M., Theoretical Aerodynamics, Dover Publications, New York, 1958.

3. Meirovitch, L., Analytical Methods in Vibrations, The Macmillan Company, London, 1967.

4. Meirovitch, L., Computational Methods in Structural Dynamics, Sijthoff-Noordhoff International Publishers, Alphen aan den Rijn, The Netherlands, 1980.

SYSTEM PARAMETERS

1. Geometric parameters:

c	6.30	ft
L	20.00	ft
y_o	2.00	ft
y_θ	0.50	ft

2. Material parameters:

m	4.65	$lb\ ft^{-2} sec^2$
I_θ	16.50	$lb\ sec^2$
EI	1.00(6)	$lb\ ft^2$
JG	1.00(7)	$lb\ ft^2$

3. Other parameters:

ρ	0.00237	$lb\ ft^{-4} sec^2$
$\frac{dc_L}{d\theta}$	2π	-

Table 1

CONSERVATIVE SYSTEM (V = 0)

case	frequency ω(rad/sec)					
	first	second	third	fourth	fifth	sixth
n=m=1	4.076	63.235*	—	—	—	—
n=m=2	4.076	25.518	63.449*	189.110	—	—
n=m=3	4.076	25.517	63.415*	71.406	190.42	315.08

$$\lambda = \pm i\omega$$

*This frequency corresponds to the mode for which the general system first reached a critical flutter condition as the velocity was increased.

Table 2

	V	n=m=1 α	n=m=1 ω	n=m=2 α	n=m=2 ω	n=m=3 α	n=m=3 ω
FIRST MODE	0	0.000	4.076	0.000	4.076	0.000	4.076
	100	-0.508	4.051	-0.508	4.051	-0.508	4.051
	200	-1.036	3.971	-1.036	3.971	-1.036	3.971
	300	-1.606	3.816	-1.607	3.816	-1.607	3.816
	400	-2.248	3.543	-2.250	3.542	-2.250	3.542
	500	-3.042	3.000	-3.005	3.040	-3.005	3.040
	600	-3.923	1.944	-3.933	1.932	-3.933	1.932
SECOND MODE	0	0.000	63.235	0.000	25.518	0.000	25.517
	100	-0.072	63.026	-0.508	25.521	-0.508	25.520
	200	-0.124	62.396	-1.019	25.530	-1.020	25.530
	300	-0.133	61.333	-1.540	25.546	-1.541	25.547
	400	-0.971	59.820	-2.074	25.570	-2.076	25.574
	500	+0.102	57.830	-2.632	25.604	-2.636	25.610
	600	+0.445	55.329	-3.228	25.647	3.234	25.657

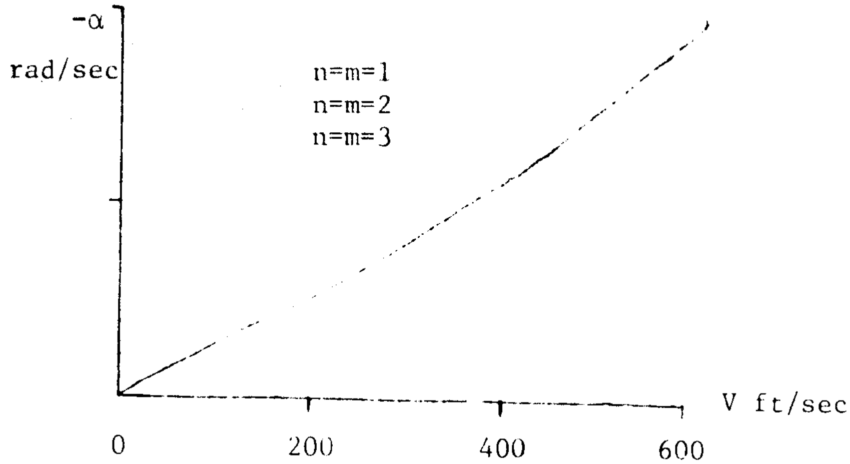
$\lambda = \alpha \pm i\omega$ rad/sec

Table 3

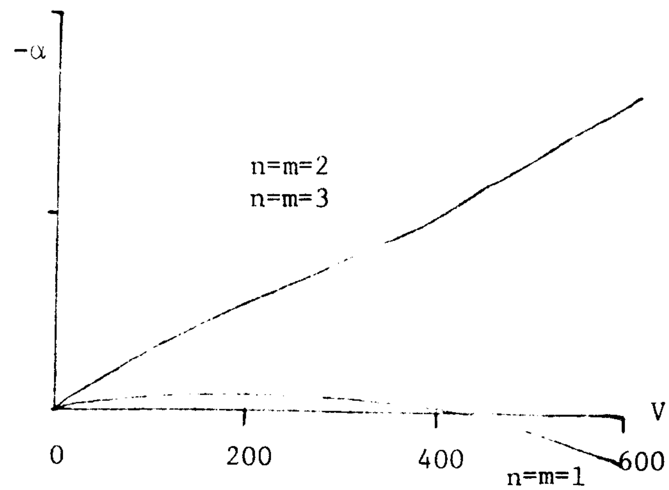
		n=m=1		n=m=2		n=m=2	
	V	α	ω	α	ω	α	ω
THIRD MODE	0	—	—	0.000	63.449	0.000	63.415
	100	—	—	-0.060	63.229	-0.064	63.199
	200	—	—	-0.098	62.567	-1.040	62.543
	300	—	—	-0.089	61.447	-0.096	61.433
	400	—	—	+0.001	59.851	-0.006	59.847
	500	—	—	+0.218	57.751	+0.212	57.752
	600	—	—	+0.636	55.098	+0.631	55.109
FOURTH MODE	0	—	—	0.000	189.110	0.000	71.406
	100	—	—	-0.086	189.040	-0.506	71.408
	200	—	—	-0.170	188.840	-0.015	71.413
	300	—	—	-0.249	188.510	-1.527	71.423
	400	—	—	-0.323	188.050	-2.043	71.442
	500	—	—	-0.389	187.450	-2.564	71.470
	600	—	—	-0.444	186.720	-3.091	71.508

$\lambda = \alpha \pm i\omega$ rad/sec

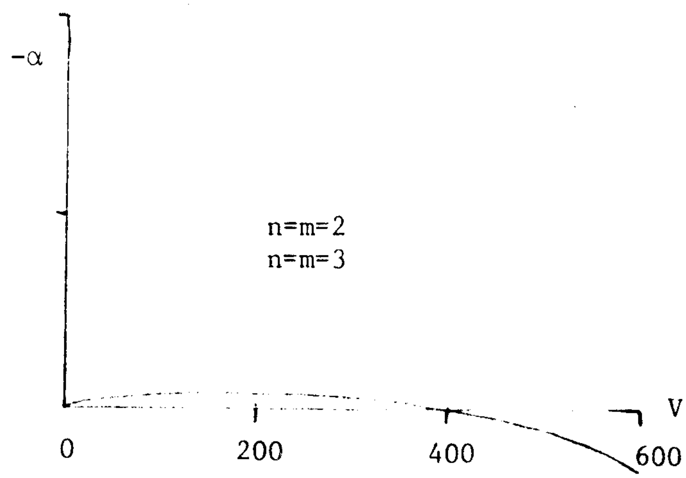
Table 4



FIRST MODE



SECOND MODE



THIRD MODE

FIGURE 3