RESPONSE OF A CIRCULAR PLATE SUBJECTED TO BASE EXCITATION
Revision C

By Tom Irvine
Email: tomirvine@aol.com

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Introduction

Consider the circular plate in Figure 1.

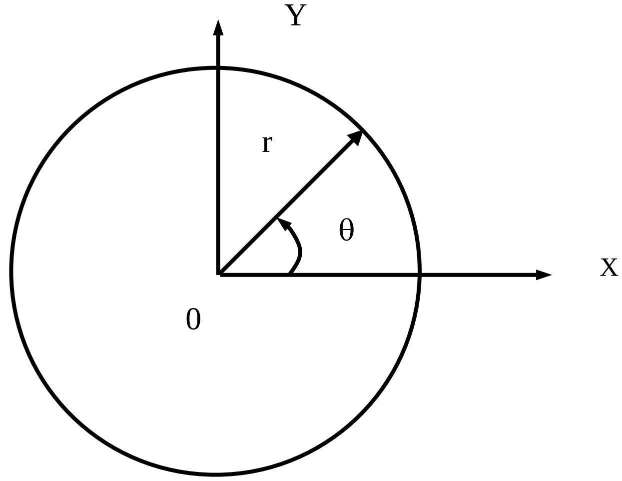

Figure 1. View Top Looking Down

Let $z(r, \theta, t)$ represent the out-of-plane displacement. The plate is subjected to uniform base excitation $\ddot{w}(t)$ as shown in Figure 2.

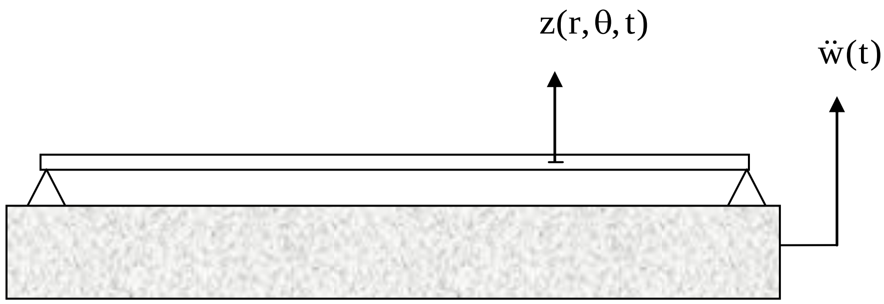

Figure 2. Side View

The equation of motion is

$$D_e \nabla^4 z(r,\theta,t) + \rho h \frac{\partial^2}{\partial t^2} z(r,\theta,t) = -\rho h \frac{d^2 w}{dt^2} \tag{1}$$

The plate stiffness factor D_e is given by

$$D_e = \frac{E h^3}{12\left(1-\mu^2\right)} \tag{2}$$

where

E	=	elastic modulus
h	=	plate thickness
μ	=	Poisson's ratio
ρ	=	mass per volume

The term on the right-hand-side of equation (1) is the inertial force per unit area.

Note that

$$\beta^4 = \frac{\omega^2 \rho h}{D_e} \tag{3}$$

where ω is the natural frequency in (radians/sec).

$$\beta = \left[\frac{\omega^2 \rho h}{D_e} \right]^{1/4} \tag{4}$$

Note the following relationship for the homogeneous problem from Reference 1.

$$\nabla^4 z(r, \theta, t) = \beta^4 z(r, \theta, t) \tag{5}$$

Substitute equation (5) into (1).

$$D_e \, \beta^4 z(r, \theta, t) + \rho h \frac{\partial^2}{\partial t^2} \, z(r, \theta, t) = -\rho h \frac{d^2 w}{dt^2} \tag{6}$$

Substitute equation (3) into (6).

$$\omega^2 \rho h z(r, \theta, t) + \rho h \frac{\partial^2}{\partial t^2} z(r, \theta, t) = -\rho h \frac{d^2 w}{dt^2} \tag{7}$$

$$\omega^2 z(r, \theta, t) + \frac{\partial^2}{\partial t^2} z(r, \theta, t) = -\frac{d^2 w}{dt^2} \tag{8}$$

The displacement is the double series of the mass-normalized mode shapes $Z_{mn}(\beta_{mn} r, \theta)$ and the time function $T_{mn}(t)$.

$$z(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Z_{mn}(\beta_{mn} r, \theta)\, T_{mn}(t) \tag{9}$$

Substitute equation (9) into (8).

$$\omega^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Z_{mn}(\beta_{mn} r, \theta)\, T_{mn}(t) + \frac{\partial^2}{\partial t^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Z_{mn}(\beta_{mn} r, \theta)\, T_{mn}(t) = -\frac{d^2 w}{dt^2} \tag{10}$$

$$\omega^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Z_{mn}(\beta_{mn} r, \theta)\, T_{mn}(t) + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Z_{mn}(\beta_{mn} r, \theta) \frac{d^2}{dt^2} T_{mn}(t) = -\frac{d^2 w}{dt^2} \tag{11}$$

Multiply each term by $Z_{uv}(\beta_{uv} r, \theta)$. Omit the arguments for brevity.

$$\omega^2 Z_{uv} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Z_{mn}\, T_{mn} + Z_{uv} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Z_{mn} \frac{d^2}{dt^2} T_{mn} = -Z_{uv} \frac{d^2 w}{dt^2} \tag{12}$$

$$\omega^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Z_{uv} Z_{mn}\, T_{mn} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Z_{uv} Z_{mn} \frac{d^2}{dt^2} T_{mn} = -Z_{uv} \frac{d^2 w}{dt^2} \tag{13}$$

Integrate with respect to area.

$$\int_0^{2\pi}\int_0^a \omega^2 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} Z_{uv} Z_{mn}\, T_{mn}\, r\,dr\,d\theta$$

$$+\int_0^{2\pi}\int_0^a \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} Z_{uv} Z_{mn}\, \frac{d^2}{dt^2} T_{mn}\, r\,dr\,d\theta = -\int_0^{2\pi}\int_0^a Z_{uv} \frac{d^2 w}{dt^2}\, r\,dr\,d\theta \tag{14}$$

$$\omega^2 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\int_0^{2\pi}\int_0^a Z_{uv} Z_{mn}\, r\,dr\,d\theta\right] T_{mn}$$

$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\int_0^{2\pi}\int_0^a Z_{uv} Z_{mn}\, r\,dr\,d\theta\right]\frac{d^2}{dt^2} T_{mn} = -\left[\int_0^{2\pi}\int_0^a Z_{uv}\, r\,dr\,d\theta\right]\frac{d^2 w}{dt^2} \tag{15}$$

The orthogonality relationship is

$$\int_0^{2\pi}\int_0^a Z_{uv} Z_{mn}\, r\,dr\,d\theta = \begin{cases} 1, \text{ for } u = m \text{ and } v = n \\ 0, \text{ for } u \neq m \text{ or } v \neq n \end{cases} \tag{16}$$

Thus

$$\omega^2 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} T_{mn} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{d^2}{dt^2} T_{mn} = -\left[\int_0^{2\pi}\int_0^a Z_{mn}\, r\,dr\,d\theta\right]\frac{d^2 w}{dt^2} \tag{17}$$

$$\omega^2 T_{mn} + \frac{d^2}{dt^2} T_{mn} = -\left[\int_0^{2\pi}\int_0^a Z_{mn} \ r\,dr\,d\theta\right]\frac{d^2 w}{dt^2} \tag{18}$$

Add a damping term. Also apply indices to the natural frequency.

$$\frac{d^2}{dt^2} T_{mn} + 2\xi_{mn}\omega_{mn}\frac{d}{dt} T_{mn} + \omega_{mn}{}^2 T_{mn} = -\left[\int_0^{2\pi}\int_0^a Z_{mn} \ r\,dr\,d\theta\right]\frac{d^2 w}{dt^2} \tag{19}$$

$$\frac{d^2}{dt^2} T_{mn} + 2\xi_{mn}\omega_{mn}\frac{d}{dt} T_{mn} + \omega_{mn}{}^2 T_{mn} = -\,\Gamma_{mn}\,\frac{d^2 w}{dt^2} \tag{20}$$

The participation factor Γ_{mn} is

$$\Gamma_{mn} = \int_0^{2\pi}\int_0^a Z_{mn} \ r\,dr\,d\theta \tag{21}$$

Take a Fourier transform of both sides of equation (20).

$$\int_{-\infty}^{\infty}\left\{\frac{d^2}{dt^2} T_{mn} + 2\xi_{mn}\omega_{mn}\frac{d}{dt} T_{mn} + \omega_{mn}{}^2 T_{mn}\right\}\exp(-j\omega t)dt = -\,\Gamma_{mn}\int_{-\infty}^{\infty}\frac{d^2 w}{dt^2}\exp(-j\omega t)dt \tag{22}$$

$$\int_{-\infty}^{\infty} \frac{d^2}{dt^2} T_{mn} \exp(-j\omega t)dt + 2\xi_{mn}\omega_{mn}\int_{-\infty}^{\infty} \frac{d}{dt} T_{mn} \exp(-j\omega t)dt$$

$$+\omega_{mn}{}^2\int_{-\infty}^{\infty} T_{mn}(t)\exp(-j\omega t)dt = -\Gamma_{mn}\int_{-\infty}^{\infty} \frac{d^2 w}{dt^2}\exp(-j\omega t)dt$$

(23)

$$\int_{-\infty}^{\infty} \frac{d}{dt} T_{mn} \exp(-j\omega t)dt = j\omega\int_{-\infty}^{\infty} T_{mn}(t) \exp(-j\omega t)dt \tag{24}$$

$$\int_{-\infty}^{\infty} \frac{d}{dt} T_{mn} \exp(-j\omega t)dt = -\omega^2\int_{-\infty}^{\infty} T_{mn}(t) \exp(-j\omega t)dt \tag{25}$$

$$-\omega^2\int_{-\infty}^{\infty} T_{mn}(t) \exp(-j\omega t)dt + j2\xi_{mn}\omega_{mn}\omega\int_{-\infty}^{\infty} T_{mn}(t) \exp(-j\omega t)dt$$

$$+\omega_{mn}{}^2\int_{-\infty}^{\infty} T_{mn}(t)\exp(-j\omega t)dt \quad = -\Gamma_{mn}\int_{-\infty}^{\infty} \ddot{w}(t)\exp(-j\omega t)dt$$

(26)

$$\left[\left(\omega_{mn}{}^2-\omega^2\right)+j2\xi_{mn}\omega_{mn}\omega\right]\int_{-\infty}^{\infty} T_{mn}(t) \exp(-j\omega t)dt = -\Gamma_{mn}\int_{-\infty}^{\infty} \ddot{w}(t)\exp(-j\omega t)dt$$

(27)

Let

$$\hat{T}_{mn}(\omega) = \int_{-\infty}^{\infty} T_{mn}(t)\exp(-j\omega t)dt \tag{28}$$

$$\ddot{W}(\omega) = \int_{-\infty}^{\infty} \ddot{w}(t)\exp(-j\omega t)dt \tag{29}$$

Substitute equations (28) and (29) into (27).

$$\left[\left(\omega_{mn}{}^{2}-\omega^{2}\right)+j2\xi_{mn}\omega_{mn}\omega\right]\hat{T}_{mn}(\omega)=-\Gamma_{mn}\ddot{W}(\omega) \tag{30}$$

$$\frac{\hat{T}_{mn}(\omega)}{W(\omega)}=-\Gamma_{mn}\left[\frac{1}{(\omega_{mn}{}^{2}-\omega^{2})+j2\xi_{mn}\omega_{mn}\omega}\right] \tag{31}$$

$$Z(\beta_{mn}r,\theta)\frac{\hat{T}_{mn}(\omega)}{W(\omega)}=-\Gamma_{mn}Z(\beta_{mn}r,\theta)\left[\frac{1}{(\omega_{mn}{}^{2}-\omega^{2})+j2\xi_{mn}\omega_{mn}\omega}\right] \tag{32}$$

The relative displacement frequency response function $\hat{Z}(r,\theta,\omega)$ is

$$\frac{\hat{Z}(r,\theta,\omega)}{W(\omega)}=-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\frac{\Gamma_{mn}Z(\beta_{mn}r,\theta)}{(\omega_{mn}{}^{2}-\omega^{2})+j2\xi_{mn}\omega_{mn}\omega}\right] \tag{33}$$

The absolute acceleration frequency response function $\ddot{U}(r,\theta,\omega)$ is

$$\frac{\ddot{U}(r,\theta,\omega)}{W(\omega)}=1+\omega^{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\frac{\Gamma_{mn}Z(\beta_{mn}r,\theta)}{(\omega_{mn}{}^{2}-\omega^{2})+j2\xi_{mn}\omega_{mn}\omega}\right] \tag{34}$$

The mode shapes are defined by

$$Z_{mn}(\beta_{mn} r, \theta) = \left[C_{mn} J_n\left(\beta_{mn} r\right) + D_{mn} I_n\left(\beta_{mn} r\right)\right]\cos(n\theta) \tag{35}$$

The mode shape for a simply supported plate is

$$Z_{mn}(\beta_{mn} r, \theta) = C_{mn}\left\{ J_n(\beta_{mn} r, \theta) - \left[\frac{J_n(\lambda)}{I_n(\lambda)}\right] I_n(\beta_{mn} r, \theta) \right\}\cos(n\theta) \tag{36}$$

where

$\beta_{mn} = \lambda / a$

a is the radius

The roots are found via

$$I_n(\lambda) J_{n+1}(\lambda) + J_n(\lambda) I_{n+1}(\lambda) = \frac{2\lambda}{1-\mu}\left[J_n(\lambda)\, I_n(\lambda)\right], \quad \text{at} \quad \lambda = \beta a \tag{37}$$

$$\rho h \int_0^a \int_0^{2\pi} \left[Z_{mn}(\beta_{mn} r, \theta)\right]^2 r \, d\theta \, dr = 1 \tag{38}$$

The coefficients C_{mn} are found via

$$\rho h \int_0^a \int_0^{2\pi} \left[C_{mn} \left\{ J_n(\beta_{mn} r, \theta) - \left[\frac{J_n(\lambda_{mn})}{I_n(\lambda_{mn})} \right] I_n(\beta_{mn} r, \theta) \right\} \cos(n\theta) \right]^2 r \, d\theta \, dr = 1$$

(39)

The participation factor Γ_{mn} is

$$\Gamma_{mn} = \rho h \int_0^a \int_0^{2\pi} C_{mn} \left\{ J_n(\beta_{mn} r) - \left[\frac{J_n(\lambda_{mn})}{I_n(\lambda_{mn})} \right] I_n(\beta_{mn} r) \right\} \cos(n\theta) \, r \, d\theta \, dr \tag{38}$$

Note that

$$\Gamma_{mn} = 0 \quad \text{for } n \geq 1 \tag{40}$$

<u>Example</u>

Consider a 48 inch diameter, 0.5 inch thick, aluminum plate that is simply supported. Each mode has a damping ratio of 0.05. The plate is subjected to uniform base excitation. Determine the frequency response functions for the center of the plate.

The calculation is performed using Matlab script: circular_SS.m

```
>> circular_SS

 circular_SS.m   ver 2.1  February 28, 2012

 by Tom Irvine   Email: tomirvine@aol.com

 This program calculates the natural frequencies of the
 bending modes of a simply-supported circular plate
 The solution is derived from Bessel functions.
 It also calculates the base excitation transfer function.
```

```
Enter unit type: 1=English 2=metric
1
Enter plate type:
1=homogeneous 2=honeycomb-sandwich
1

Enter diameter (inch) 48


Enter thickness (inch) 0.5

Enter material:
1=aluminum  2=steel  3=G10  4=other 1

Structural mass =    90.48 lbm

Add uniformly distributed non-structural mass ?
1=yes  2=no
2

    Structural mass =    90.48 lbm
Non-structural mass =        0 lbm
         Total mass =    90.48 lbm
    Total thickness =      0.5 in
           Diameter =       48 in
             Volume =    904.8 in^3
Overall mass density =      0.1 lbm/in^3

Plate Stiffness Factor  D = 1.145e+005 lbf in
                      D/E =  0.01145 in^3
```

Natural Frequency (Hz)	n	k	C	D	root	PF	EMM
40.54	0	0	1.94464	0.036856	2.2215	0.8419	0.7087
114.16	1	0	2.48779	0.005541	3.7280	0.0000	0.0000
210.39	2	0	2.94641	0.001373	5.0610	0.0000	0.0000
244.49	0	1	2.94015	-0.000536	5.4558	-0.3489	0.1217
328.21	3	0	3.35403	0.000419	6.3212	0.0000	0.0000
398.62	1	1	3.33271	-0.000098	6.9663	0.0000	0.0000
466.89	4	0	3.72722	0.000144	7.5393	0.0000	0.0000
575.94	2	1	3.68533	-0.000025	8.3736	0.0000	0.0000
609.72	0	2	3.68423	0.000014	8.6157	0.2247	0.0505
625.93	5	0	4.07507	0.000053	8.7294	0.0000	0.0000
776.62	3	1	4.00971	-0.000007	9.7236	0.0000	0.0000
844.36	1	2	4.00499	0.000003	10.1389	0.0000	0.0000
1000.39	4	1	4.31286	-0.000002	11.0359	0.0000	0.0000
1103.11	2	2	4.30233	0.000001	11.5887	0.0000	0.0000

```
   n = nodal diameters
   k = nodal circles
  PF = participation factor
 EMM = effective modal mass

 plot mode shapes?
 1=yes 2=no
 1

 Calculate Base Excitation Transfer function?  1=yes  2=no
 1

 Enter uniform damping ratio: 0.05

      Enter starting freq (Hz): 1
        Enter ending freq (Hz): 2000
 Enter the frequency step (Hz): 0.1


  Maximum Transfer Magnitude Values

Relative Displacement =  0.09394 in/G at    40.4 Hz
Absolute Acceleration =     15.8 G/G at    40.5 Hz
```

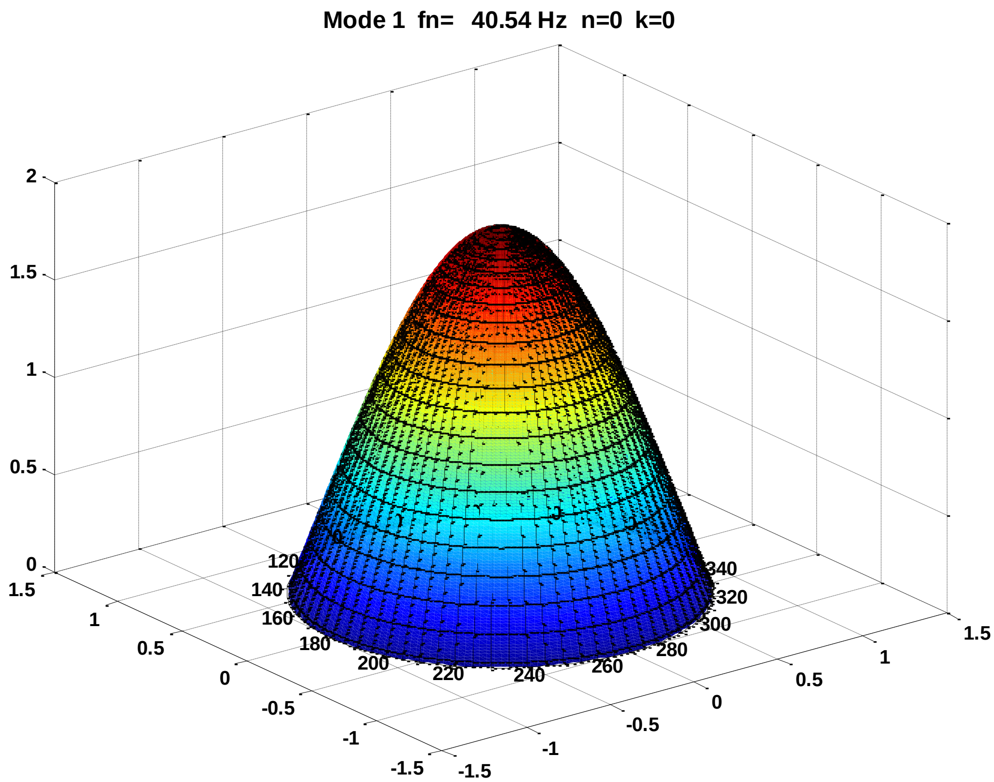
Mode 1 fn=   40.54 Hz   n=0 k=0
2
1.5
1
0.5
0
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5
120
140
160
180
200
220
240
260
280
300
320
340

Figure 3.

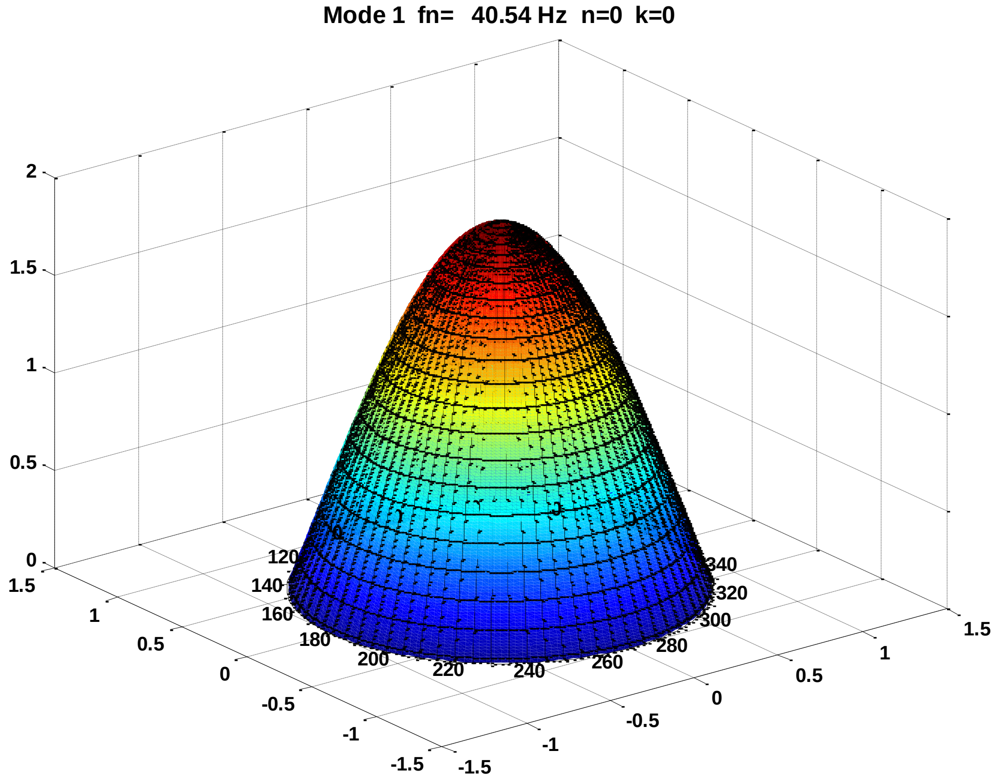

Figure 4.

The fundamental mode is excited by uniform base excitation.

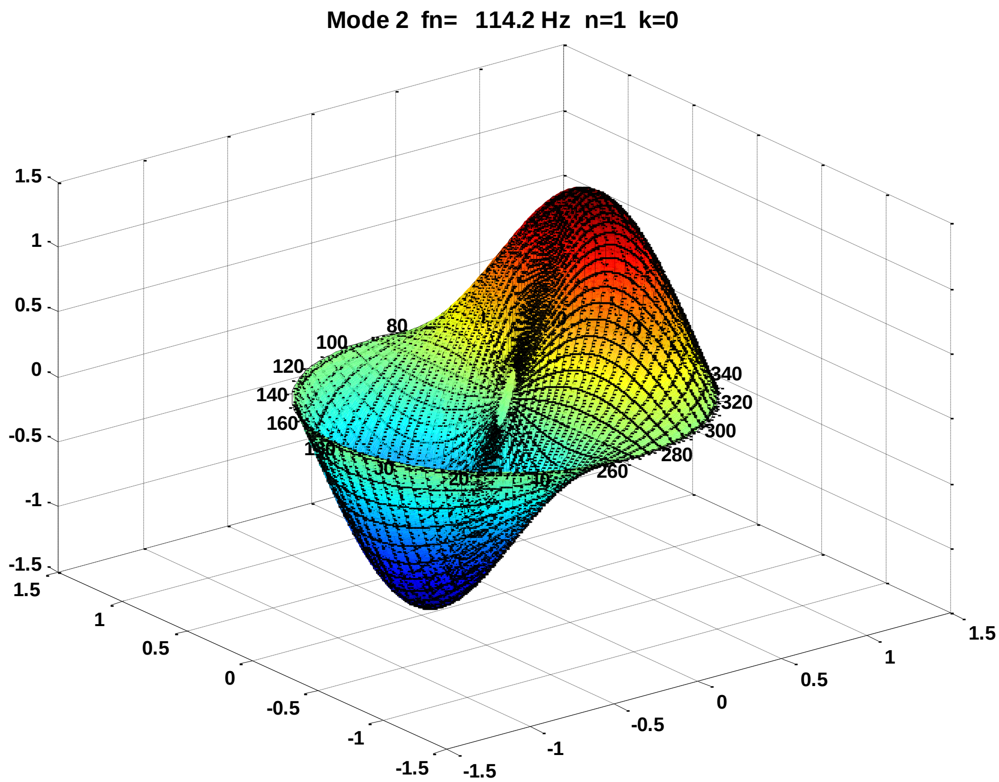

Figure 5.

This mode is shown for reference only. It is not excited by uniform base excitation.

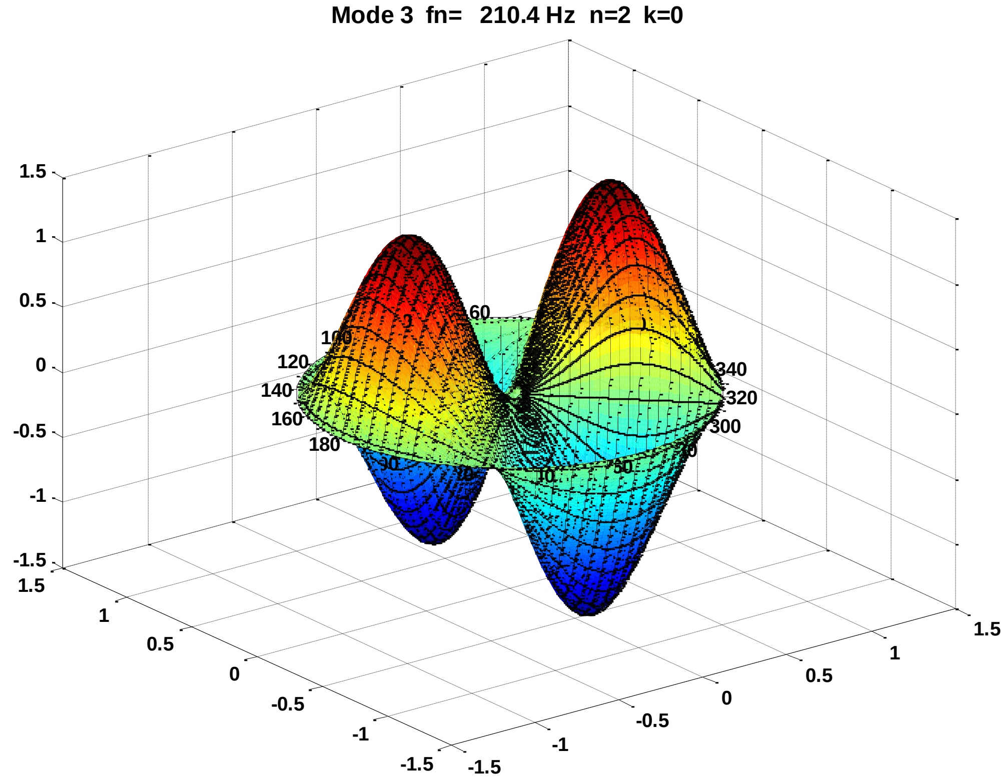

Figure 6.

This mode is shown for reference only. It is not excited by uniform base excitation.

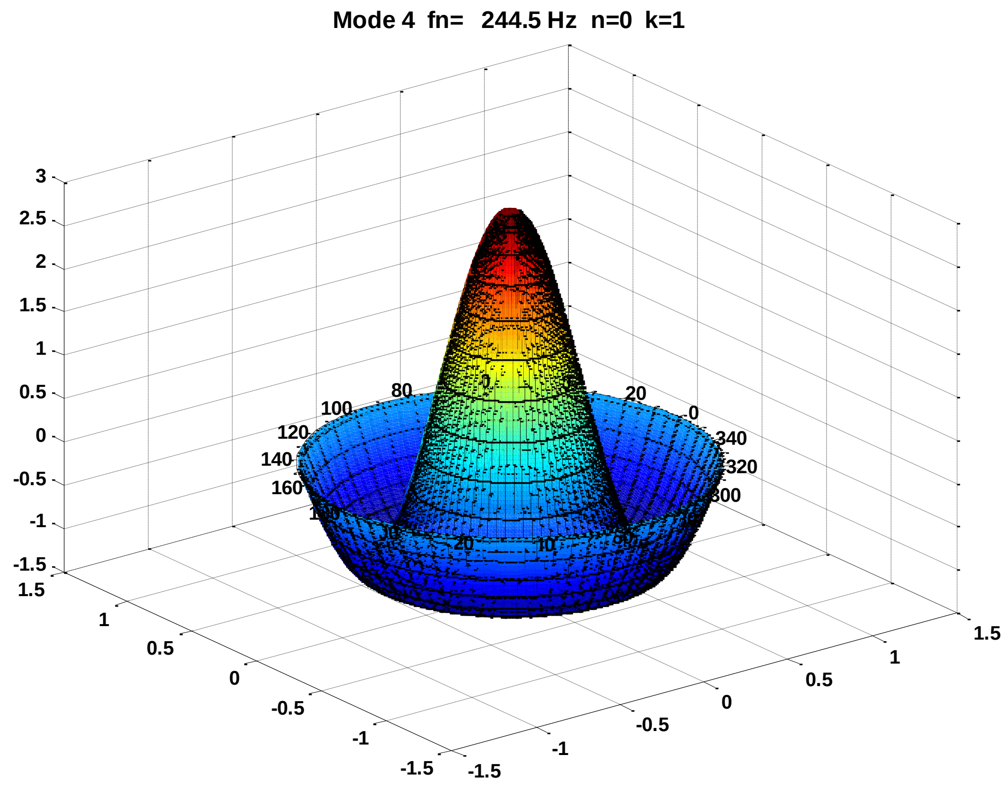

Figure 7.

The fourth mode is excited by uniform base excitation.

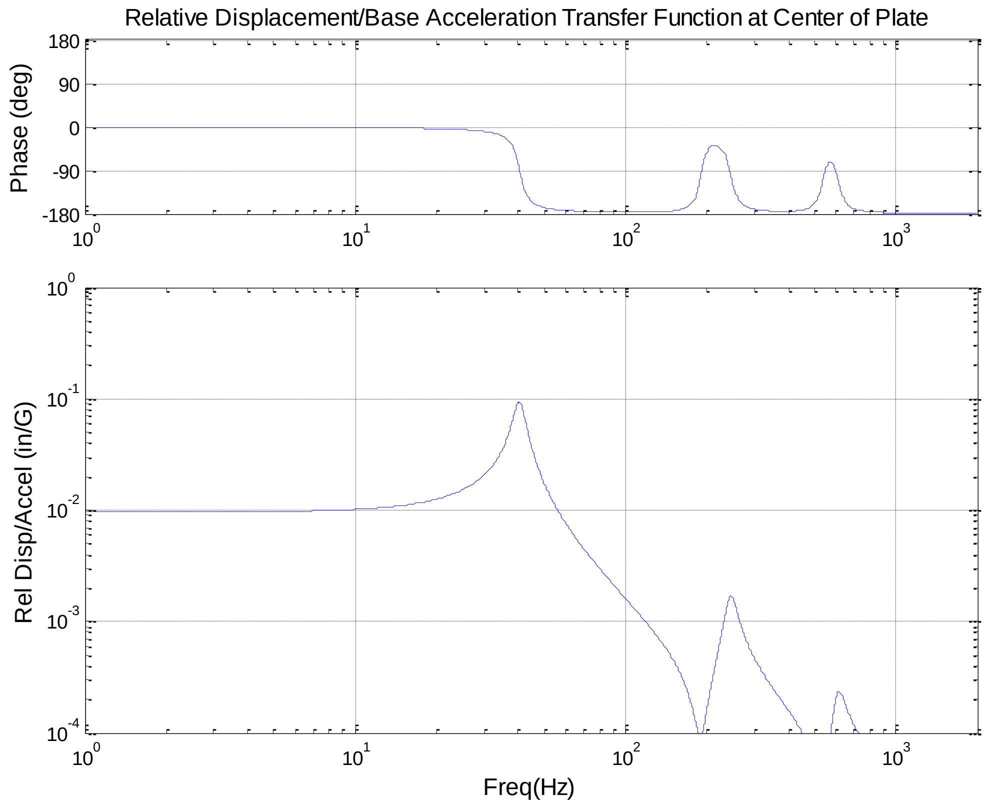

Figure 8.

Maximum Transfer Magnitude:

Relative Displacement = 0.09394 in/G at 40.4 Hz

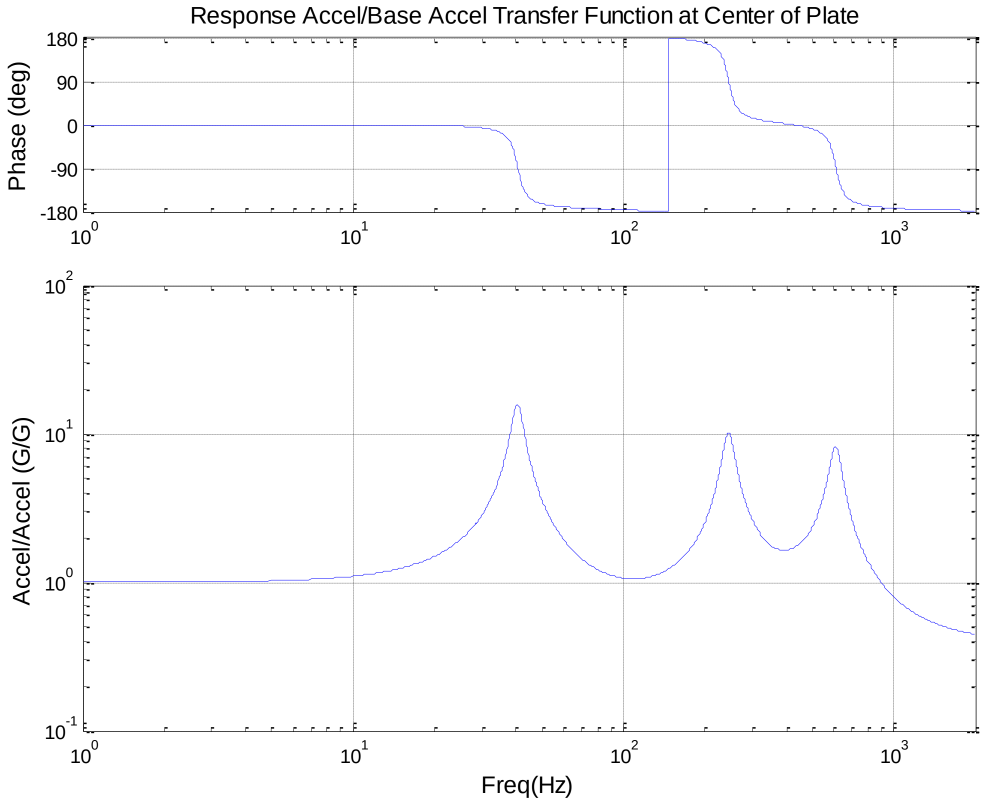

Figure 9.

Maximum Transfer Magnitude:

Absolute Acceleration = 15.8 G/G at 40.5 Hz

Reference

1. T. Irvine, Natural Frequencies of Circular Plate Bending Modes, Revision F, Vibrationdata, 2012.