THE NATURAL FREQUENCY OF A CIRCULAR PLATE WITH FOUR FIXED POINT SUPPORTS

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Introduction

The Rayleigh method is used in this tutorial to determine the fundamental bending frequency. The method is taken from References 1 through 3.

A displacement function is assumed for the Rayleigh method which satisfies the geometric boundary conditions. The assumed displacement function is substituted into the strain and kinetic energy equations.

The Rayleigh method gives a natural frequency that is an upper limit of the true natural frequency. The method would give the exact natural frequency if the true displacement function were used. The true displacement function is called an eigenfunction.

Consider the circular plate in Figure 1.

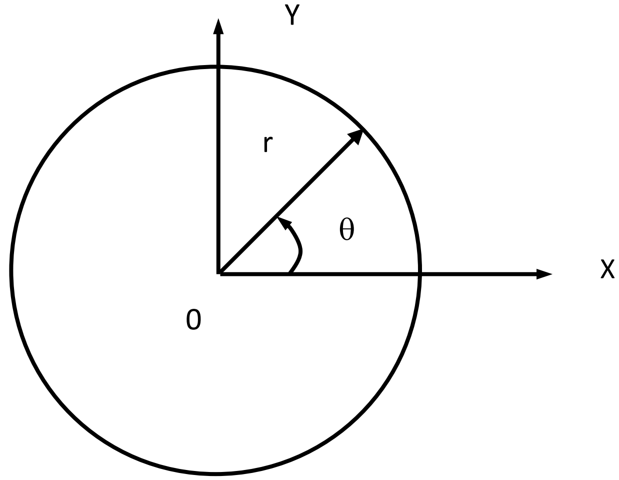


Figure 1.

The plate is fixed at four points along the circumference, spaced 90 degrees apart.

Let Z represent the out-of-plane displacement.

The total strain energy V of the plate is

$$V = \frac{D_e}{2}\int_0^{2\pi}\int_0^{R}\left[\left(\frac{\partial^2 Z}{\partial r^2}+\frac{1}{r}\frac{\partial Z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 Z}{\partial \theta^2}\right)^2 - 2(1-\mu)\frac{\partial^2 Z}{\partial^2 r}\left(\frac{1}{r}\frac{\partial Z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 Z}{\partial \theta^2}\right)\right.$$

$$\left. + 2(1-\mu)\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial Z}{\partial \theta}\right)\right\}^2\right] r\,dr\,d\theta \tag{1}$$

Note that the plate stiffness factor D_e is given by

$$D_e = \frac{Eh^3}{12\left(1-\mu^2\right)} \tag{2}$$

where

E = elastic modulus

H = plate thickness

μ = Poisson's ratio

The total kinetic energy T of the plate bending is given by

$$T = \frac{\rho h \Omega^2}{2}\int_0^{2\pi}\int_0^{R} Z^2 \; r\,dr\,d\theta \tag{3}$$

where

ρ = mass per volume

Ω = angular natural frequency

The bending moment M_r at the circumference is

$$M_r = -D_e \left[\frac{\partial^2 Z}{\partial r^2} + \mu \left(\frac{1}{r} \frac{\partial Z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 Z}{\partial r^2} \right) \right] \tag{4}$$

The shear V_r at the circumference is

$$V_r = Q_r + \frac{1}{r} \frac{\partial M_{r\theta}}{\partial \theta} \tag{5}$$

where

$$Q_r = -D_e \frac{\partial}{\partial r} \left(\nabla^2 Z \right) \tag{6}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \tag{7}$$

A displacement mode shape was derived after much trial-and-error. The mode shape satisfies the condition of zero displacement at the four fixed points. It approximately satisfies the conditions of zero moment and zero shear with respect to the radius at the other locations along the circumference.

The estimated unscaled displacement mode shape is

$$Z(r,\theta) = B \left[a \left(\frac{r}{R} \right)^u + b \left(\frac{r}{R} \right)^w \right] \cos(n\theta) + C + \sum_{i=1}^{4} A_i \cos\left(\alpha_i \pi \frac{r}{R} \right) \tag{8}$$

where

B	=	0.8744
a	=	2.8808
b	=	-1.8808
u	=	3.6059
W	=	3.9948
n	=	4
C	=	2.3869

A_i	α_i
1.9619	0.8709
-1.3048	0.0846
-3.4583	0.6709
3.0771	0.6157

The corresponding natural frequency fn from the Rayleigh method is

$$fn \approx \frac{6.075}{2\pi R^2}\sqrt{\frac{D_e}{\rho h}} \tag{9}$$

Example

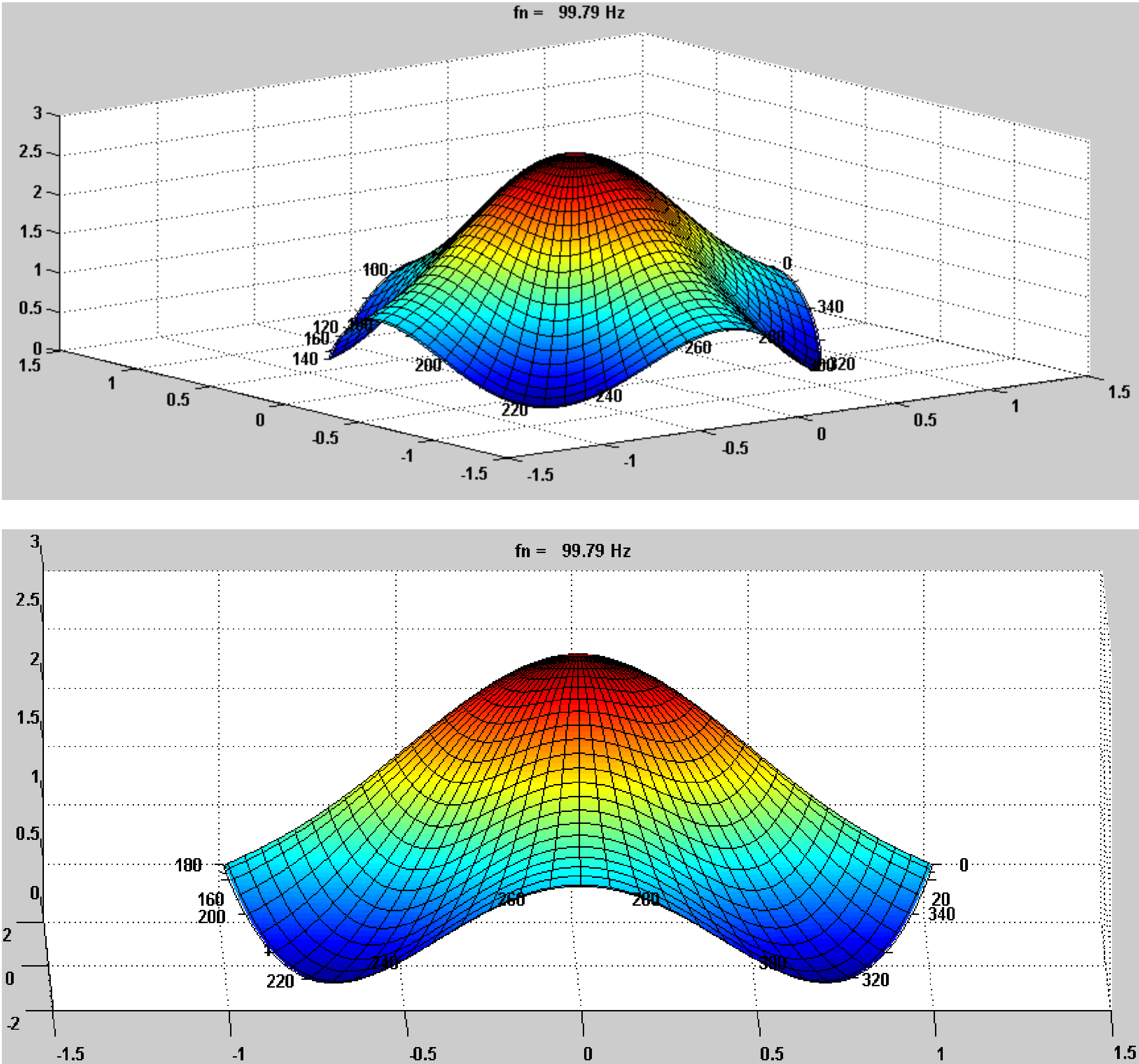


Figure 2. Fundamental Bending Mode, fn = 99.8 Hz

A 48-in diameter, 1-in thick alumimum plated is fixed at four discreted points. The estimated mode shape from equation (8) is plotted in Figure 2. The radius is normalized in the plot so that its maximum value is one.

Note that the natural frequencies for a simply-supported and fixed-fixed plate are 82 and 168 Hz, respectively, for a plate with the same material and dimensions.

The Rayleigh method results in Figure 2 are similar to those for the finite element model in Appendix A.

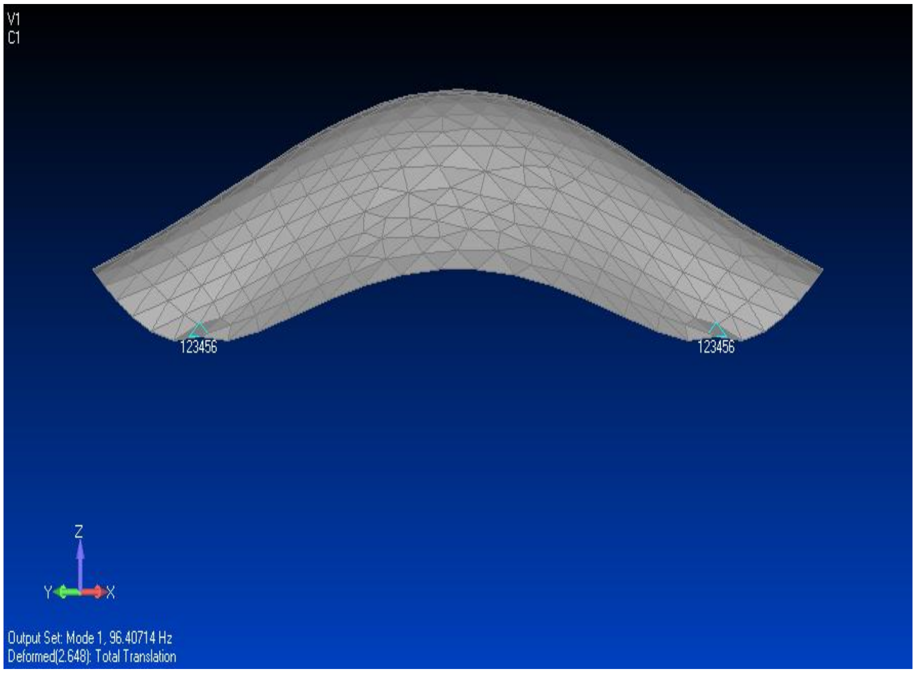
Some further work can be done to improve the estimated mode shape, but the problem is challenging.

References

1. Dave Steinberg, Vibration Analysis for Electronic Equipment, Wiley-Interscience, New York, 1988.
2. Weaver, Timoshenko, and Young; Vibration Problems in Engineering, Wiley-Interscience, New York, 1990.
3. Arthur W. Leissa, Vibration of Plates, NASA SP-160, National Aeronautics and Space Administration, Washington D.C., 1969.
4. Jan Tuma, Engineering Mathematics Handbook, McGraw-Hill, New York, 1979.
5. L. Meirovitch, Analytical Methods in Vibrations, Macmillan, New York, 1967.
6. W. Soedel, Vibrations of Shells and Plates, Third Edition, Marcel Dekker, New York, 2004.

APPENDIX A

Finite Element Model



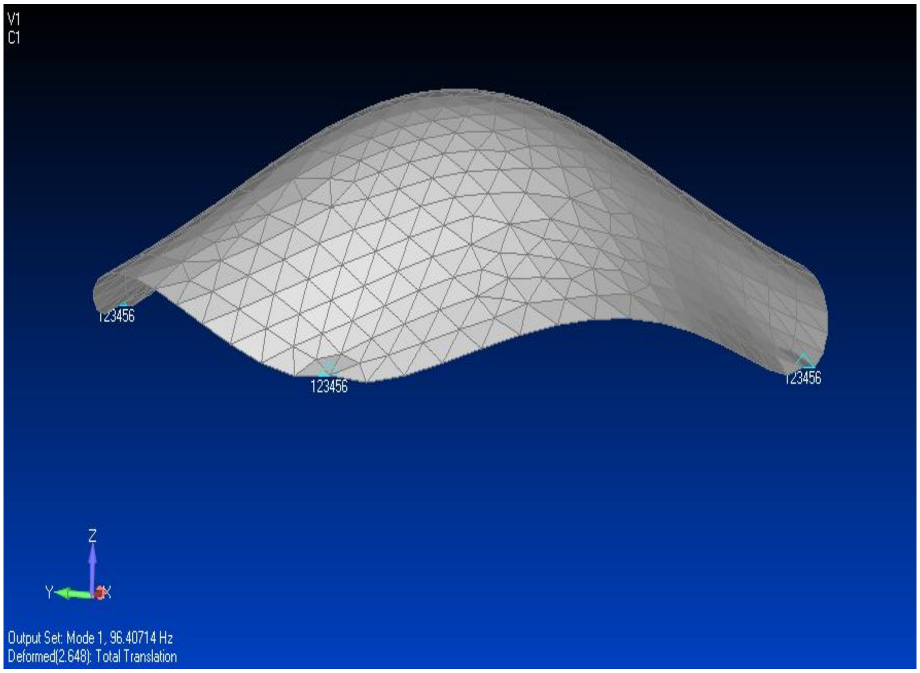


Figure A-1. Fundamental Bending Mode, fn = 96.4 Hz

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