Coloumb damping is dry friction damping. Consider the free vibration response of a single-degree-of-freedom system subjected to Coloumb damping.

The damping force $F$ is

$$F = \mu mg$$

where

- $\mu$ = friction coefficient
- $m$ = Mass
- $g$ = acceleration of gravity

Assume that the friction coefficient is constant for simplicity.
The governing equation of motion for the displacement $x$ is

$$m \ddot{x} + k x = -F \text{sgn}(\dot{x})$$

(2)

where

$$k = \text{Stiffness}$$

The $\text{sgn}(\dot{x})$ function represents the sign of $\dot{x}$.

As an alternative, the governing equation can be written as

$$m \ddot{x} + k x = -F \frac{\dot{x}}{|\dot{x}|}$$

(3)

The governing equation is solved in a piecewise-linear manner.

Assume that initial displacement $x(0)$ is

$$x(0) > \frac{F}{k}$$

(4)

Also assume that the initial velocity is zero.

Consider the equation of motion for negative velocity.

$$m \ddot{x} + k x = F \quad \text{for} \quad \dot{x} < 0$$

(5)

$$\ddot{x} + \left(\frac{k}{m}\right)x = \frac{F}{m}$$

(6)

$$\omega_n^2 = \left(\frac{k}{m}\right)$$

(7)

$$E = \left(\frac{F}{m}\right)$$

(8)
\[
\ddot{x} + \omega_n^2 x = E
\] (9)

The equation is solved using Laplace transforms.

\[
L\{\ddot{x} + \omega_n^2 x\} = L\{E\}
\] (10)

\[
s^2 X(s) - sx(0) + \omega_n^2 X(s) = \frac{E}{s}
\] (11)

\[
\left[s^2 + \omega_n^2\right]X(s) - sx(0) = \frac{E}{s}
\] (12)

\[
X(s) = \frac{E}{s + \omega_n^2} + \frac{sx(0)}{s^2 + \omega_n^2}
\] (13)

Take the inverse Laplace transform.

\[
x(t) = x(0)\cos \omega_n t + \frac{E}{\omega_n^2} \left[1 - \cos \omega_n t\right]
\] (15)

\[
x(t) = \frac{E}{\omega_n^2} \left[x(0) - \frac{E}{\omega_n^2}\right] \cos \omega_n t
\] (16)

\[
x(t) = \frac{F}{m\omega_n^2} + \left[x(0) - \frac{F}{m\omega_n^2}\right] \cos \omega_n t
\] (17)
\[ x(t) = \frac{F}{k} + \left[ x(0) - \frac{F}{k} \right] \cos \omega_n t \quad \text{for} \quad \dot{x} < 0 \quad (18) \]

\[ \dot{x}(t) = -\omega_n \left[ x(0) - \frac{F}{k} \right] \sin \omega_n t \quad \sin \omega_n t > 0 \quad (19) \]

The velocity equals zero at

\[ t = \frac{\pi}{\omega_n} \quad (20) \]

The displacement at this time is

\[ x \left( \frac{\pi}{\omega_n} \right) = \frac{F}{k} + \left[ -x(0) + \frac{F}{k} \right] \quad (21) \]

\[ x \left( \frac{\pi}{\omega_n} \right) = \frac{2F}{k} - x(0) \quad (22) \]

Consider the equation of motion for positive velocity.

\[ m \ddot{x} + k x = -F \quad \text{for} \quad \dot{x} > 0 \quad (23) \]

The initial displacement term must be reset to the last displacement for negative velocity. Furthermore, a phase angle must be added to the argument in the cosine term.
\begin{align}
    x(t) &= -\frac{F}{k} + \left(\frac{2F}{k} - x(0)\right) + \frac{F}{k} \cos(\omega_n t + \pi) \quad \text{for } \dot{x} > 0 \quad (24) \\
    x(t) &= -\frac{F}{k} + \left(\frac{3F}{k} - x(0)\right) \cos(\omega_n t + \pi) \quad \text{for } \dot{x} > 0 \quad (25) \\
    \dot{x}(t) &= -\omega_n \left[\frac{3F}{k} - x(0)\right] \sin(\omega_n t + \pi) \quad (26)
\end{align}

The first negative displacement peak thus has an amplitude that is 2 F/k less than the initial displacement in terms of absolute values. This reduction factor can also be derived from the work-energy relationship in Appendix A.

The pattern continues such that the envelope has a linear decay.

The velocity returns to zero for

\[ t = \frac{2\pi}{\omega_n} \quad (27) \]

\[ x\left(\frac{2\pi}{\omega_n}\right) = -\frac{F}{k} \left[\frac{3F}{k} - x(0)\right] \quad (28) \]

\[ x\left(\frac{2\pi}{\omega_n}\right) = x(0) - \frac{4F}{k} \quad (29) \]

Each consecutive positive peak is thus 4 F/k lower than the previous positive peak.

The process is then repeated.
**Example**

A single-degree-of-freedom system has

mass = 1 kg

stiffness = 20,000 N/m

friction coefficient = 0.4

initial displacement = 5 mm

The resulting displacement is shown in Figure 2.

The displacement converges to $\frac{F}{k}$, where $F = \mu mg$.

Depending on the initial displacement, the displacement may also converge to $-\frac{F}{k}$. 
Figure 2.

References


APPENDIX A

Energy Method

The potential energy is set equal to the work done by friction for one cycle.

\[
\frac{1}{2} k (x_1^2 - x_2^2) = F (x_1 + x_2)
\]  \hspace{1cm} (A-1)

where \( x_1 \) and \( x_2 \) are consecutive positive peaks.

Note that the kinetic energy is zero at the instantaneous time that each peak occurs.

The work-energy relationship is satisfied if

\[
F = \frac{1}{2} k (x_1 - x_2)
\]  \hspace{1cm} (A-2)

\[
(x_1 - x_2) = \frac{2F}{k}
\]  \hspace{1cm} (A-3)
Matlab Script

disp(' '); disp(' dry.m   ver 1.0   June 25, 2005 '); disp(' by Tom Irvine   Email: tomirvine@aol.com '); disp(' '); disp(' This program calculates the response of a '); disp(' single-degree-of-freedom system subjected to dry damping '); disp(' '); clear all; disp(' Enter mass (kg) ') m=input(' '); disp(' Enter stiffness (N/m) ') k=input(' '); disp(' Enter coefficient of friction ') mu=input(' '); disp(' Enter initial displacement (mm) ') xo=input(' '); xo=xo/1000.; disp(' '); F=mu*m*(9.81); fk=F/k; disp(' '); omegan=sqrt(k/m); fn=omegan/(2.*pi); disp(' '); out1=sprintf('
 fn = %8.4g Hz
',fn); disp(out1); disp(' '); out1=sprintf(' F/k = %8.4g mm
',(F/k)*1000.); disp(out1); disp(' '); if( F/k > xo ) disp(' '); disp(' No oscillation. '); disp(' F/k > xo '); end disp(' '); T=1/fn; disp(' '); dt = T/100.; delta=2.*pi/100; disp(' '); num=12.*T/dt; j=1;
tdelay=0.;
arg=0.;
for(i=1:(num+1))
    t(i)=(i-1)*dt;
    
% arg=arg+delta;
if(arg>2.*pi)
    arg=arg-2.*pi;
end

% if(arg>=0 && arg<=pi)
    if( (xo-fk) <=0)
        x(i)=xo;
    else
        x(i)= fk + ( xo - fk )*cos(arg);
    end
    x1=x(i);
else
    if( abs(x1) < fk )
        x(i)=x1;
    else
        x(i)= -fk +( x1 + fk )*cos(arg+pi);
    end
    xo=x(i);
end

end
x=x*1000.;
plot(t,x);
xlabel(' Time(sec) ');
ylabel(' Displacement(mm) ');
out1=sprintf(' SDOF Response  Dry Friction   fn=%8.4g Hz ',fn);
title(out1);
grid on;