THE ENERGY METHOD   Revision C

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June 5, 2002

Introduction

Dynamic systems can be characterized in terms of one or more natural frequencies. The natural frequency is the frequency at which the system would vibrate if it were given an initial disturbance and then allowed to vibrate freely.

There are many available methods for determining the natural frequency. Some examples are

1. Newton's Law of Motion
2. Rayleigh's Method
3. Energy Method
4. Lagrange's Equation

Note that the Rayleigh, Energy, and Lagrange methods are closely related.

Some of these methods directly yield the natural frequency. Others yield a governing equation of motion, from which the natural frequency may be determined.

This tutorial focuses on the energy method, which is an example of a method which yields an equation of motion.

Definition of the Energy Method

The total energy of a conservative system is constant. Thus,

$$\frac{d}{dt}\left(KE + PE\right) = 0 \tag{1}$$

where

KE = kinetic energy
PE = potential energy

Kinetic energy is the energy of motion, as calculated from the velocity.

Potential energy has several forms. One is strain energy. Another is the work done against a gravity field.

Pendulum Example

Consider the pendulum shown in Figure A-1.

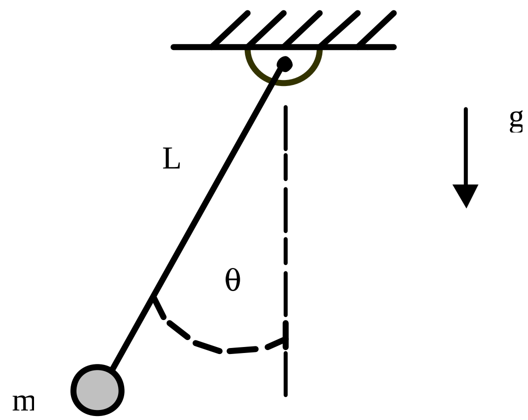

Figure A-1.

The potential energy is

$$PE = mgL\,(1 - \cos\theta) \qquad \text{(A-1)}$$

The kinetic energy is

$$KE = \frac{1}{2} m (L\dot{\theta})^2 \qquad \text{(A-2)}$$

Apply the energy method to the pendulum example, using equations (1), (A-1), and (A-2).

$$\frac{d}{dt}\left[\frac{1}{2} m (L\dot{\theta})^2 + mgL\,(1 - \cos\theta)\right] = 0 \qquad \text{(A-3)}$$

$$m(L^2\,\dot{\theta})\,\ddot{\theta} + mg\,L(\sin\theta)\,\dot{\theta} = 0 \qquad \text{(A-4)}$$

Divide through by $m(L^2\,\dot{\theta})$.

$$\ddot{\theta} + \frac{g}{L}(\sin\theta) = 0 \qquad \text{(A-5)}$$

For small angles,

$$\sin\theta \approx \theta \tag{A-6}$$

Thus, the equation of motion is

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \tag{A-7}$$

Again, assume a displacement of

$$\theta(t) = \alpha \sin(\omega_n t) \tag{A-8}$$

The velocity equation is

$$\dot{\theta}(t) = \alpha\,\omega_n \cos(\omega_n t) \tag{A-9}$$

The acceleration equation is

$$\ddot{\theta}(t) = -\alpha\omega_n{}^2 \sin(\omega_n t) \tag{A-10}$$

Substitute into equations (A-10) and (A-8) into (A-7).

$$-\alpha\omega_n{}^2 \sin(\omega_n t) + \left[\frac{g}{L}\right]\alpha \sin(\omega_n t) = 0 \tag{A-11}$$

$$-\alpha\omega_n{}^2 \sin(\omega_n t) + \left[\frac{g}{L}\right]\alpha \sin(\omega_n t) = 0 \tag{A-12}$$

$$-\omega_n{}^2 + \left[\frac{g}{L}\right] = 0 \tag{A-13}$$

$$\omega_n{}^2 = \left[\frac{g}{L}\right] \tag{A-14}$$

The pendulum natural frequency is

$$\omega_n = \sqrt{\frac{g}{L}} \tag{A-15}$$

This exercise has demonstrated that the equation of motion can be represented as

$$\ddot{\theta} + \omega_n^{\,2}\,\theta = 0 \tag{A-16}$$

<u>Cantilever Beam with End Mass</u>

Consider a mass mounted on the end of a cantilever beam, as shown in Figure B-1. Assume that the end-mass is much greater than the mass of the beam.

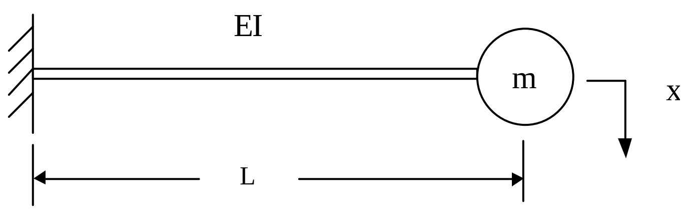

Figure B-1.

E is the modulus of elasticity.
I is the area moment of inertia.
L is the length.
g is gravity.
m is the mass,
x is the displacement.

The static stiffness at the end of the beam is

$$k = \frac{3EI}{L^3} \tag{B-1}$$

Equation (B-1) is derived in Reference 1.

The potential energy is

$$PE = \frac{1}{2}\left[\frac{3EI}{L^3}\right]x^2 \tag{B-2}$$

The kinetic energy is

$$KE = \frac{1}{2} m\,\dot{x}^2 \tag{B-3}$$

Apply the energy method,

$$\frac{d}{dt}\left\{\frac{1}{2}m\dot{x}^2+\frac{1}{2}\left[\frac{3EI}{L^3}\right]x^2\right\}=0 \qquad \text{(B-4)}$$

$$m\dot{x}\ddot{x}+\left[\frac{3EI}{L^3}\right]x\dot{x}=0 \qquad \text{(B-5)}$$

Divide through by the velocity term.

$$m\ddot{x}+\left[\frac{3EI}{L^3}\right]x=0 \qquad \text{(B-6)}$$

Divide through by mass. The equation of motion is

$$\ddot{x}+\left[\frac{3EI}{mL^3}\right]x=0 \qquad \text{(B-7)}$$

The governing equation of motion for simple harmonic systems is known to take the form of

$$\ddot{x}+\omega_n{}^2\ x=0 \qquad \text{(B-8)}$$

Note that equation (B-8) was demonstrated in the pendulum example.

The natural frequency of the end mass supported by the cantilever beam is thus

$$\omega_n{}^2=\left[\frac{3EI}{mL^3}\right] \qquad \text{(B-9)}$$

$$\omega_n=\sqrt{\frac{3EI}{mL^3}} \qquad \text{(B-10)}$$

Spring-Mass System including the Mass of the Spring

Consider the system in Figure C-1.

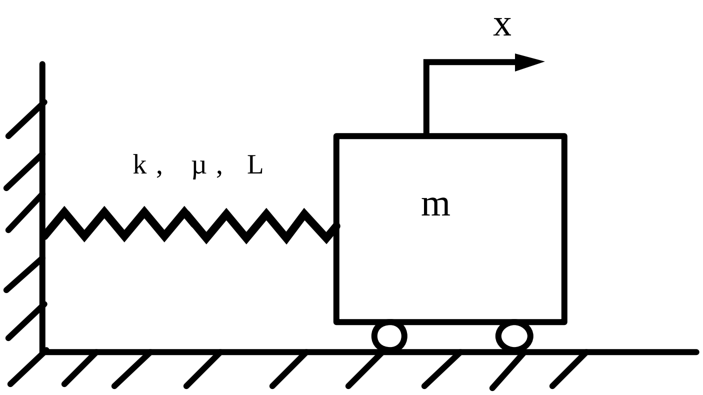

Figure C-1.

The variables are

m is the block mass,
k is the spring stiffness,
μ is mass density (mass/length) of the spring,
L is the length of the spring,
x is the absolute displacement of the mass.

The potential energy of the spring is

$$PE = \frac{1}{2}kx^2 \quad \text{(C-1)}$$

The kinetic energy of the block is

$$KE_{block} = \frac{1}{2}m\dot{x}^2 \quad \text{(C-2)}$$

The kinetic energy of the spring is found in the following steps. Define a local variable ξ which is a measure of the distance along the spring.

$$0 \leq \xi \leq L \quad \text{(C-3)}$$

The velocity at any point on the spring is thus

$$\dot{x}\,\frac{\xi}{L} \tag{C-4}$$

Now divide the spring into n segments. The kinetic energy of the spring is thus

$$KE_{spring} = \frac{1}{2}\sum_{i=1}^{n}\left\{\left[\dot{x}\,\frac{\xi}{L}\right]^2 \mu\,\Delta\xi\right\} \tag{C-5}$$

Take the limit as n approaches infinity.

$$KE_{spring} = \frac{1}{2}\int_0^L \left[\dot{x}\,\frac{\xi}{L}\right]^2 \mu\,d\xi \tag{C-6}$$

$$KE_{spring} = \frac{1}{6}\mu\left[\frac{\dot{x}}{L}\right]^2 \xi^3\Big|_0^L \tag{C-7}$$

$$KE_{spring} = \frac{1}{6}\mu\left[\frac{\dot{x}}{L}\right]^2 L^3 \tag{C-8}$$

$$KE_{spring} = \frac{1}{6}\mu\,\dot{x}^2\,L \tag{C-9}$$

The total kinetic energy is thus

$$KE = \frac{1}{2}m\dot{x}^2 + \frac{1}{6}\mu\,\dot{x}^2\,L \tag{C-10}$$

Take the derivative of the energy terms.

$$\frac{d}{dt}\left\{\frac{1}{2}m\dot{x}^2 + \frac{1}{6}\mu L\dot{x}^2 + \frac{1}{2}kx^2\right\} = 0 \tag{C-11}$$

$$m\dot{x}\ddot{x} + \frac{1}{3}\mu L\dot{x}\ddot{x} + kx\dot{x} = 0 \tag{C-12}$$

$$m\ddot{x} + \frac{1}{3}\mu L\ddot{x} + kx = 0 \qquad \text{(C-13)}$$

$$\left[m + \frac{1}{3}\mu L\right]\ddot{x} + kx = 0 \qquad \text{(C-14)}$$

$$\ddot{x} + \left[\frac{k}{m + \frac{1}{3}\mu L}\right]x = 0 \qquad \text{(C-15)}$$

Again, the governing equation of motion for simple harmonic systems is known to take the form of

$$\ddot{x} + \omega_n{}^2\, x = 0 \qquad \text{(C-16)}$$

The natural frequency is thus

$$\omega_n = \sqrt{\frac{k}{m + \frac{1}{3}\mu L}} \qquad \text{(C-17)}$$

<u>Spring-Mass System with Pendulum</u>

The energy method may also be applied to systems with multiple degrees-of-freedom.

Consider the previous example with an added pendulum, as shown in Figure D-1.

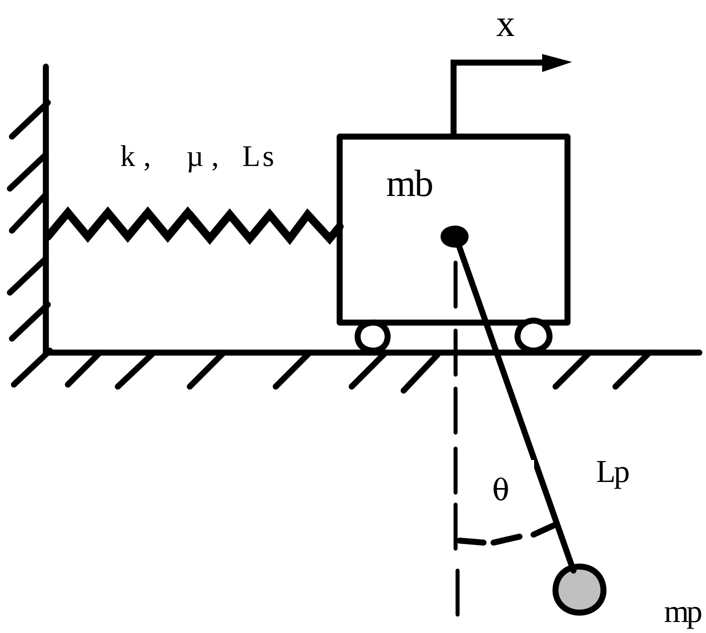

Figure D-1.

The variables are

mb	is the block mass,
k	is the spring stiffness,
μ	is mass density (mass/length) of the spring,
Ls	is the length of the spring,
mp	is the pendulum mass,
Lp	is the pendulum length,
x	is the absolute displacement of the mass,
θ	is the pendulum angular displacement.

The total potential energy is

$$PE = \frac{1}{2}kx^2 + mp\ g\ Lp\ (1 - \cos\theta) \qquad \text{(D-1)}$$

The total kinetic energy is

$$KE = \frac{1}{2}mb\ \dot{x}^2 + \frac{1}{6}\mu\ \dot{x}^2\ Ls + \frac{1}{2}mp\ (Lp\ \dot{\theta} + \dot{x})^2 \qquad \text{(D-2)}$$

Apply the energy method.

$$\frac{d}{dt}\left\{\frac{1}{2}mb\ \dot{x}^2 + \frac{1}{6}\mu\,\dot{x}^2\ Ls\ + \frac{1}{2}mp\ (Lp\,\dot{\theta}+\dot{x})^2 + \frac{1}{2}kx^2 + mp\ g\ Lp\ (1-\cos\theta)\right\} = 0$$

(D-3)

$$mb\ \dot{x}\ddot{x} + \frac{1}{3}\mu\,\dot{x}\ddot{x}\ Ls\ + mp\ (Lp\,\dot{\theta}+\dot{x})(Lp\,\ddot{\theta}+\ddot{x}) + kx\ \dot{x} + mp\ g\ Lp\ (\sin\theta)\dot{\theta} = 0$$

(D-4)

$$mb\ \dot{x}\ddot{x} + \frac{1}{3}\mu\,\dot{x}\ddot{x}\ Ls\ + mp\,Lp^2\,\dot{\theta}\ddot{\theta}\ + mp\,Lp\,\dot{\theta}\ddot{x} + mp\,Lp\ \ddot{\theta}\dot{x} + \ mp\ \dot{x}\ddot{x}$$
$$+ kx\ \dot{x} + mp\ g\ Lp\ (\sin\theta)\,\dot{\theta} = 0$$

(D-5)

For small angles,

$$\sin\theta \approx \theta \qquad \text{(D-6)}$$

Thus,

$$mb\ \dot{x}\ddot{x} + \frac{1}{3}\mu\,\dot{x}\ddot{x}\ Ls\ + mp\,Lp^2\,\dot{\theta}\ddot{\theta}\ + mp\,Lp\,\dot{\theta}\ddot{x} + mp\,Lp\ \ddot{\theta}\dot{x} + \ mp\ \dot{x}\ddot{x}$$
$$+ kx\ \dot{x} + mp\ g\ Lp\ \theta\ \dot{\theta} = 0$$

(D-7)

$$+\left\{mb\ \ddot{x} + \frac{1}{3}\mu\ \ddot{x}\ Ls + mp\,Lp\ \ddot{\theta} + \ mp\ \ddot{x} + kx\ \right\}\dot{x}$$
$$+\left\{mp\,Lp^2\ \ddot{\theta}\ + mp\,Lp\ \ddot{x} + mp\ g\ Lp\ \theta\ \right\}\dot{\theta}\ =\ 0$$

(D-8)

Equation (D-8) can be separated into two equations

$$\left\{ mb\ \ddot{x} + \frac{1}{3}\mu\ \ddot{x}\ Ls + mp\,Lp\ \ddot{\theta} + \ \ mp\ \ddot{x} + kx \right\}\dot{x} = 0$$

(D-9)

And

$$\left\{ mp\,Lp^2\ \ddot{\theta}\ + mp\,Lp\ \ddot{x} + mp\ g\ Lp\ \theta \right\}\dot{\theta} = 0$$

(D-10)

Divide each equation by its respect velocity term.

$$\left\{ mb\ \ddot{x} + \frac{1}{3}\mu\ \ddot{x}\ Ls + mp\,Lp\ \ddot{\theta} + \ \ mp\ \ddot{x} + kx \right\} = 0 \qquad \text{(D-11a)}$$

$$\left\{ \left[mb + mp + \frac{1}{3}\mu\ \ Ls \right] \ddot{x} + \ + mp\,Lp\ \ddot{\theta}\ \ + kx \right\} = 0 \qquad \text{(D-11b)}$$

And

$$\left\{ mp\,Lp^2\ \ddot{\theta}\ + mp\,Lp\ \ddot{x} + mp\ g\ Lp\ \theta \right\} = 0 \qquad \text{(D-12a)}$$

$$\left\{ mp\,Lp\ \ddot{\theta}\ + mp\ \ \ddot{x} + mp\ g\ \ \theta \right\} = 0 \qquad \text{(D-12b)}$$

$$\left\{ Lp\ \ddot{\theta}\ +\ \ \ddot{x} + g\ \ \theta \right\} = 0 \qquad \text{(D-12c)}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} \left[mb + mp + \frac{1}{3}\mu\ \ Ls \right] & mp \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta}\,Lp \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & g \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(D-13)

Alternate Form

Again, the energy method can be used to derive the equation of motion. The following form can be used if the natural frequency is the only parameter of interest.

$$KE_1 + PE_1 = KE_2 + PE_2 \quad \text{(E-1)}$$

The subscripts represent time.

Conclusion

The energy method is suitable for reasonably simple systems.

The energy method may be inappropriate for complex systems, however. The reason is that the distribution of the vibration amplitude is required before the kinetic energy equation can be derived. Prior knowledge of the "mode shapes" is thus required.

The Lagrange method is better suited for complex systems, as discussed in Reference 2.

References

1. T. Irvine, Natural Frequencies of Beam Bending Modes, Revision E, Vibrationdata.com Publications, 1998.

2. T. Irvine, Lagrange's Equation, Vibrationdata.com Publications, 1999.