ESCAPE VELOCITY Revision A

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Introduction

A body of constant mass is projected upward from the Earth's surface with an initial velocity. The escape velocity is the smallest initial velocity for which the body will not return to the Earth.

Derivation via the Force Method

Assume no air resistance. Account for the variation of the Earth's gravitational field with altitude. The free-body diagram is shown in Figure 1.



Figure 1.

Let

m = mass of the body g = acceleration of gravity R = radius of the Earth x = altitude above the surface v = velocity $v_o = initial velocity$ $v_f = final velocity$ The sum of the forces equals the linear momentum per Newton's law.

$$\sum F = m \frac{dv}{dt}$$
(1)

The momentum in the direction away from the Earth is equal to the negative gravitational force.

$$m\frac{dv}{dt} = -mg\frac{R^2}{(x+R)^2}$$
(2)

Divide through by m.

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g \frac{R^2}{(x+R)^2} \tag{3}$$

Apply the chain rule.

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\mathbf{t}} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\mathbf{x}}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\mathbf{t}} \tag{4}$$

Recall

$$v = \frac{dx}{dt}$$
(5)

Substitute equation (5) into (4).

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\mathbf{t}} = \mathbf{v}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\mathbf{x}} \tag{6}$$

Substitute equation (6) into (3).

$$v\frac{dv}{dx} = -g\frac{R^2}{(x+R)^2}$$
(7)

Multiply through by dx.

$$v dv = -g \frac{R^2}{(x+R)^2} dx$$
(8)

Integrate over the distance from zero to infinity.

$$\int_{v_0}^{v_f} v \, dv = -\int_0^\infty g \frac{R^2}{(x+R)^2} \, dx$$
(9)

$$\frac{1}{2}v^2 \Big|_{v_0}^{v_f} = g \frac{R^2}{(x+R)} \Big|_0^{\infty}$$
(10)

$$\frac{1}{2} \left[v_{\rm f}^2 - v_{\rm o}^2 \right] = -g \frac{R^2}{R} \tag{11}$$

$$\frac{1}{2} \left[v_{\rm f}^2 - v_{\rm o}^2 \right] = -g \, R \tag{12}$$

Set the final velocity equal to zero.

$$-\frac{1}{2}v_0^2 = -gR$$
 (13)

$$v_0^2 = 2gR \tag{13}$$

$$v_0 = \sqrt{2gR} \tag{14}$$

Note that the initial velocity is also the escape velocity.

Derivation via the Energy Method

Equate the change in kinetic energy to the work performed against gravity.

$$\frac{1}{2}mv_{f}^{2} - \frac{1}{2}mv_{o}^{2} = -\int_{0}^{\infty}mg\frac{R^{2}}{(x+R)^{2}}dx$$
(15)

Divide through by mass

$$\frac{1}{2}v_{f}^{2} - \frac{1}{2}v_{o}^{2} = -\int_{0}^{\infty} g \frac{R^{2}}{(x+R)^{2}} dx$$
(16)

Recall that the final velocity is zero.

$$-\frac{1}{2}v_0^2 = -\int_0^\infty g \frac{R^2}{(x+R)^2} dx$$
(17)

$$-\frac{1}{2}v_0^2 = -g\frac{R^2}{(x+R)}\Big|_0^\infty$$
(18)

$$-\frac{1}{2}v_0^2 = -g\frac{R^2}{R}$$
 (19)

$$-\frac{1}{2}v_0^2 = -gR$$
 (20)

$$v_0^2 = 2gR \tag{21}$$

$$v_{O} = \sqrt{2gR}$$
(22)

Note that Equation (22) is the same as (14).

Examples

Sample escape velocities are given in Table 1.

Table 1. Escape Velocities			
Body	Radius (km)	Surface Gravitational Acceleration (m/sec ²)	Escape Velocity (km/sec)
Sun	6.96(10 ⁵)	2774	1965
Earth	6378	9.81	11.2
Moon	1738	1.67	2.41
Mars	3394	3.73	5.03