# BENDING FREQUENCY OF A FIXED-FIXED BEAM VIA THE RAYLEIGH METHOD

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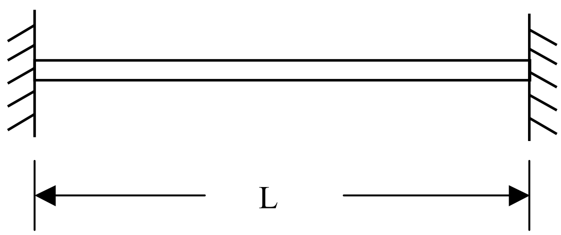
Consider a fixed-fixed beam.



| | |
|---|---|
| E | is the modulus of elasticity |
| I | is the area moment of inertia |
| L | is the length |
| ρ | is the mass density (mass/length) |

Assume a displacement function using a polynomial. Scale the independent variable so that L = 1.

$$W(\xi) = a + b\xi + c\xi^2 + d\xi^3 + e\xi^4 \quad (1)$$

$$\frac{d}{d\xi} W(\xi) = b + 2c\xi + 3d\xi^2 + 4e\xi^3 \quad (2)$$

The geometric boundary conditions are

$$W(0) = 0 \quad (3)$$

$$\frac{d}{d\xi} W(0) = 0 \tag{4}$$

$$W(L) = 0 \tag{5}$$

$$\frac{d}{d\xi} W(0) = 0 \tag{6}$$

Also, set the midpoint amplitude equal to 1.

$$W(0.5)=1 \tag{7}$$

The boundary conditions in equations (3) and (4) readily prove that a=0 and b=0, respectively.

The boundary condition in equation (5) shows that

$$c + d + e = 0 \tag{8}$$

The boundary condition in equation (6) shows that

$$2c + 3d + 4e = 0 \tag{9}$$

Equation (6) shows that

$$0.25c + 0.125d + 0.0625e = 0 \tag{10}$$

Equations (8), (9), and (10) can be arranged in matrix form.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 0.25 & 0.125 & 0.0625 \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{11}$$

The solution is

$$\begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 16 \\ -32 \\ 16 \end{bmatrix} \tag{12}$$

The displacement function is

$$W(\xi) = 16\xi^2 - 32\xi^3 + 16\xi^4 \tag{13}$$

$$\frac{d}{d\xi} W(\xi) = 32\xi - 96\xi^2 + 64\xi^3 \tag{14}$$

$$\frac{d^2}{d\xi^2} W(\xi) = 32 - 192\xi + 192\xi^2 \tag{15}$$

Now consider the independent variable and its derivatives.

$$\xi = x / L \tag{16}$$

$$d\xi = dx / L \tag{17}$$

$$dx = L d\xi \tag{18}$$

$$\frac{d}{dx} y = \frac{d}{d\xi} \frac{d\xi}{dx} y = \frac{1}{L} \frac{d}{d\xi} y \tag{19}$$

$$\frac{d^2}{dx^2} y = \frac{d^2}{d\xi^2} \frac{d^2\xi}{dx^2} y = \frac{1}{L^2} \frac{d^2}{d\xi^2} y \tag{20}$$

The Rayleigh method is used to find the natural frequency. The total potential energy and the total kinetic energy must be determined.

The total potential energy P in the beam is

$$P = \frac{EI}{2} \int_0^L \left( \frac{d^2 y}{dx^2} \right)^2 dx \tag{21}$$

$$P = \frac{EI}{2L^3} \int_0^1 \left( \frac{d^2 W}{d\xi^2} \right)^2 d\xi \tag{22}$$

$$P = \frac{EI}{2L^3} \int_0^1 \left( 32 - 192\xi + 192\xi^2 \right)^2 d\xi \tag{23}$$

The integral is evaluated using the Matlab script in Appendix B.

$$P = \frac{EI}{2L^3} (204.8) \tag{24}$$

$$P = 102.4 \frac{EI}{L^3} \tag{25}$$

The total kinetic energy T is

$$T = \frac{1}{2} \rho\ \omega_n^2 \int_0^L [y]^2 dx \tag{26}$$

$$T = \frac{1}{2} \rho \omega_n^2 L \int_0^1 [W]^2 d\xi \tag{27}$$

$$T = \frac{1}{2} \rho \omega_n^2 L \int_0^1 \left[ 16\xi^2 - 32\xi^3 + 16\xi^4 \right]^2 d\xi \tag{28}$$

$$T = \frac{1}{2} \rho \omega_n^2 L (0.406) \tag{29}$$

$$T = 0.203 \rho \omega_n^2 L \tag{30}$$

Set the maximum kinetic energy equal to the maximum potential energy.

$$0.203\,\rho\,\omega_n^2\,L = 102.4\,\frac{EI}{L^3} \tag{31}$$

$$\omega_n^2 = 504.0\,\frac{EI}{L^4\rho} \tag{32}$$

The Rayleigh method thus yields a natural frequency of

$$\omega_n = \frac{22.4}{L^2}\sqrt{\frac{EI}{\rho}} \tag{33}$$

Note that theoretical value from Reference 1 is

$$\omega_n = \left[\frac{22.373}{L^2}\right]\sqrt{\frac{EI}{\rho}} \tag{34}$$

The analysis is repeating using a trigonometric displacement function as shown in Appendix A. Both displacement functions are shown in Figure 1, normalized to an amplitude of one.

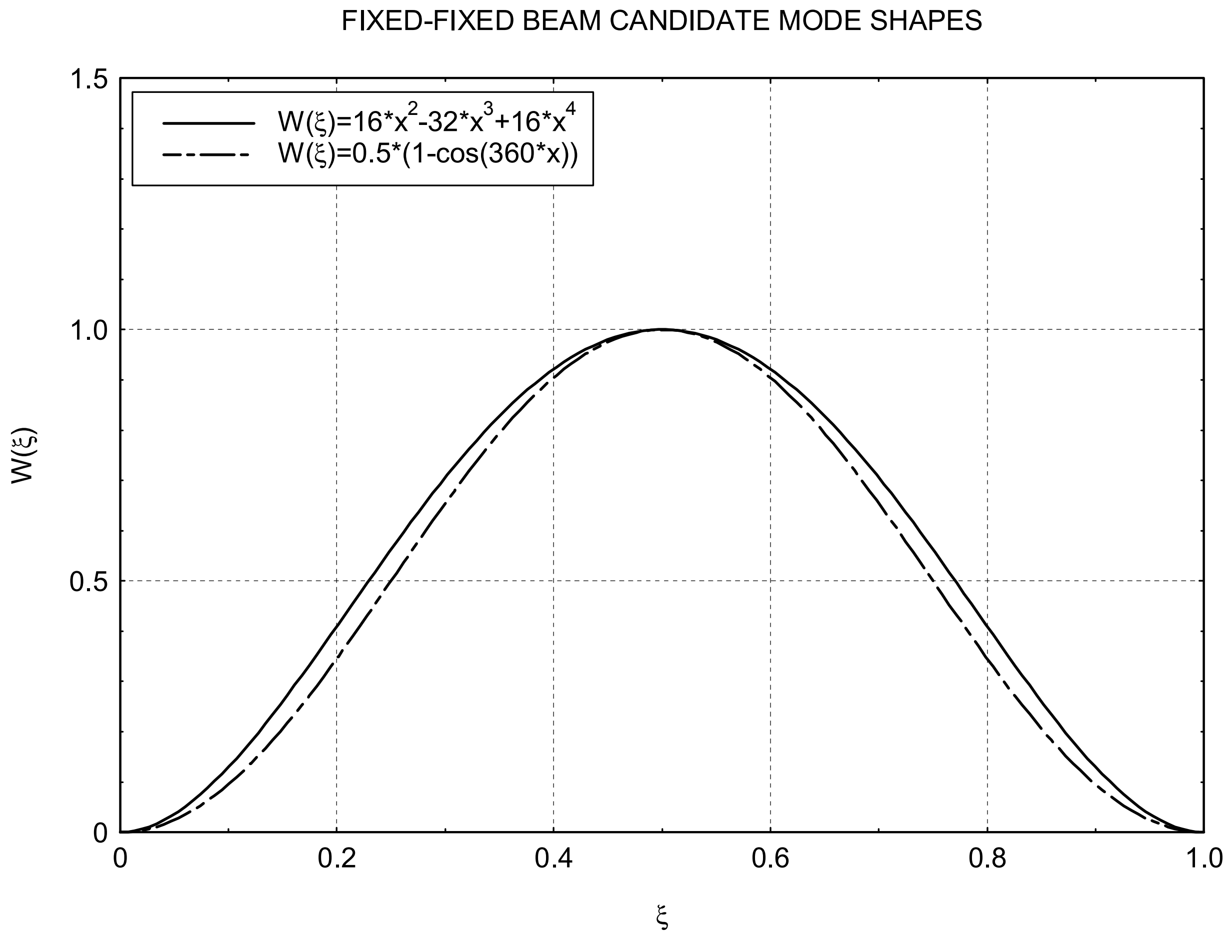



Figure 1.

<u>Reference</u>

1. T. Irvine, Bending Frequencies of Beams, Rods, and Pipes, Revision K, Vibrationdata, 2004.

# APPENDIX A

Trigonometric Displacement Function

A displacement function which satisfies the four geometric boundary conditions is

$$y(\xi) = 1 - \cos(2\pi\xi) \qquad \text{(A-1)}$$

$$\frac{d}{d\xi} y(\xi) = 2\pi \sin(2\pi\xi) \qquad \text{(A-2)}$$

$$\frac{d}{d\xi^2}^{2} y(\xi) = \left(4\pi^2\right)\cos(2\pi\xi) \qquad \text{(A-3)}$$

Note the amplitude scale factor for equation (1) is arbitrary.

The total potential energy is

$$P = \frac{EI}{2L^3} \int_0^1 \left(\left(4\pi^2\right)\cos(2\pi\xi)\right)^2 d\xi \qquad \text{(A-4)}$$

$$P = \left(16\pi^4\right)\frac{EI}{2L^3} \int_0^1 (\cos(2\pi\xi))^2 \, d\xi \qquad \text{(A-5)}$$

$$P = \left(16\pi^4\right)\frac{EI}{2L^3} \int_0^1 \left[\frac{1}{2} + \frac{1}{2}\cos(4\pi\xi)\right] d\xi \qquad \text{(A-6)}$$

$$P = \left(16\pi^4\right)\frac{EI}{2L^3} \left[\frac{1}{2}\xi + \frac{1}{8\pi}\sin(4\pi)\right]\Bigg|_0^1 \qquad \text{(A-7)}$$

$$P = \left(16\pi^4\right)\frac{EI}{2L^3}\left[\frac{1}{2}\right] \tag{A-8}$$

$$P = \frac{4\pi^4 EI}{L^3} \tag{A-9}$$

The total kinetic energy is

$$T = \frac{1}{2}\rho\,\omega_n^2\,L\int_0^1 [1-\cos(2\pi\xi)]^2\,d\xi \tag{A-10}$$

$$T = \frac{1}{2}\rho\,\omega_n^2\,L\int_0^1 \left[1-2\cos(2\pi\xi)+\cos^2(2\pi\xi)\right]d\xi \tag{A-11}$$

$$T = \frac{1}{2}\rho\,\omega_n^2\,L\int_0^1 \left[1-2\cos(2\pi\xi)+\frac{1}{2}+\frac{1}{2}\cos(4\pi\xi)\right]d\xi \tag{A-12}$$

$$T = \frac{1}{2}\rho\,\omega_n^2\,L\int_0^1 \left[\frac{3}{2}-2\cos(2\pi\xi)+\frac{1}{2}\cos(4\pi\xi)\right]d\xi \tag{A-13}$$

$$T = \frac{1}{2}\rho\,\omega_n^2\,L\left[\frac{3\xi}{2}-\frac{1}{\pi}\sin(2\pi\xi)+\frac{1}{8\pi}\sin(4\pi\xi)\right]\Bigg|_0^1 \tag{A-14}$$

$$T = \frac{3}{4}\rho\,\omega_n^2\,L \tag{A-15}$$

Equate the maximum kinetic energy with the maximum potential energy.

$$\frac{3}{4}\rho\,\omega_n^2\,L = \frac{4\pi^4 EI}{L^3} \tag{A-16}$$

$$\omega_n^2 = \frac{16\pi^4}{3}\frac{EI}{L^4\rho} \tag{A-17}$$

The Rayleigh method thus yields a natural frequency of

$$\omega_n = \frac{22.8}{L^2}\sqrt{\frac{EI}{\rho}} \tag{A-18}$$

Again, the theoretical value from Reference 1 is

$$\omega_n = \left[\frac{22.373}{L^2}\right]\sqrt{\frac{EI}{\rho}} \tag{A-19}$$

# APPENDIX B

Matlab Script for Polynomial Analysis

```
clear x;
clear P;
clear T;
%
P = @(x)((32-192*x+192*x.^2).^2);
QP = quad(P,0,1)
%
T=@(x)((16*x.^2-32*x.^3+16*x.^4).^2);
QT = quad(T,0,1)
%
b=QP/QT;
out5 = sprintf(' QP/QT= %8.2f \n',b);
disp(out5)
%
a=sqrt(QP/QT);
%
disp('');
out5 = sprintf(' omegan = %6.1f [sqrt(EI/rho)]/L^2 \n',a);
disp(out5)
```