Consider a fixed-fixed beam.

\[ E \] is the modulus of elasticity
\[ I \] is the area moment of inertia
\[ L \] is the length
\[ \rho \] is the mass density (mass/length)

Assume a displacement function using a polynomial. Scale the independent variable so that \( L = 1 \).

\[ W(\xi) = a + b\xi + c\xi^2 + d\xi^3 + e\xi^4 \] \hspace{1cm} (1)

\[ \frac{d}{d\xi} W(\xi) = b + 2c\xi + 3d\xi^2 + 4e\xi^3 \] \hspace{1cm} (2)

The geometric boundary conditions are

\[ W(0) = 0 \] \hspace{1cm} (3)
\[
\frac{d}{d\xi} W(0) = 0 \quad (4)
\]

\[
W(L) = 0 \quad (5)
\]

\[
\frac{d}{d\xi} W(0) = 0 \quad (6)
\]

Also, set the midpoint amplitude equal to 1.

\[
W(0.5) = 1 \quad (7)
\]

The boundary conditions in equations (3) and (4) readily prove that \(a=0\) and \(b=0\), respectively.

The boundary condition in equation (5) shows that

\[
c + d + e = 0 \quad (8)
\]

The boundary condition in equation (6) shows that

\[
2c + 3d + 4e = 0 \quad (9)
\]

Equation (6) shows that

\[
0.25c + 0.125d + 0.0625e = 0 \quad (10)
\]

Equations (8), (9), and (10) can be arranged in matrix form.

\[
\begin{bmatrix}
1 & 1 & 1 \\
2 & 3 & 4 \\
0.25 & 0.125 & 0.0625
\end{bmatrix}
\begin{bmatrix}
c \\
d \\
e
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\quad (11)
\]

The solution is

\[
\begin{bmatrix}
c \\
d \\
e
\end{bmatrix}
= 
\begin{bmatrix}
16 \\
-32 \\
16
\end{bmatrix}
\quad (12)
\]
The displacement function is

\[ W(\xi) = 16\xi^2 - 32\xi^3 + 16\xi^4 \]  \hspace{1cm} (13)

\[ \frac{d}{d\xi} W(\xi) = 32\xi - 96\xi^2 + 64\xi^3 \]  \hspace{1cm} (14)

\[ \frac{d^2}{d\xi^2} W(\xi) = 32 - 192\xi + 192\xi^2 \]  \hspace{1cm} (15)

Now consider the independent variable and its derivatives.

\[ \xi = x / L \]  \hspace{1cm} (16)

\[ d\xi = dx / L \]  \hspace{1cm} (17)

\[ dx = Ld\xi \]  \hspace{1cm} (18)

\[ \frac{d}{dx} y = \frac{d}{d\xi} \frac{dx}{d\xi} y = \frac{1}{L} \frac{d}{d\xi} y \]  \hspace{1cm} (19)

\[ \frac{d^2}{dx^2} y = \frac{d^2}{d\xi^2} \frac{d^2\xi}{dx^2} y = \frac{1}{L^2} \frac{d^2}{d\xi^2} y \]  \hspace{1cm} (20)

The Rayleigh method is used to find the natural frequency. The total potential energy and the total kinetic energy must be determined.

The total potential energy \( P \) in the beam is

\[ P = \frac{EI}{2} \int_0^L \left( \frac{d^2 y}{dx^2} \right)^2 dx \]  \hspace{1cm} (21)
\[ P = \frac{EI}{2L^3} \int_0^1 \left( \frac{d^2 W}{d\xi^2} \right)^2 \, d\xi \]  
(22)

\[ P = \frac{EI}{2L^3} \int_0^1 \left( 32 - 192\xi + 192\xi^2 \right)^2 \, d\xi \]  
(23)

The integral is evaluated using the Matlab script in Appendix B.

\[ P = \frac{EI}{2L^3} (204.8) \]  
(24)

\[ P = 102.4 \frac{EI}{L^3} \]  
(25)

The total kinetic energy \( T \) is

\[ T = \frac{1}{2} \rho \omega_n^2 \int_0^1 \left[ y \right]^2 \, dx \]  
(26)

\[ T = \frac{1}{2} \rho \omega_n^2 L \int_0^1 \left[ W \right]^2 \, d\xi \]  
(27)

\[ T = \frac{1}{2} \rho \omega_n^2 L \int_0^1 \left[ 16\xi^2 - 32\xi^3 + 16\xi^4 \right]^2 \, d\xi \]  
(28)

\[ T = \frac{1}{2} \rho \omega_n^2 L (0.406) \]  
(29)

\[ T = 0.203 \rho \omega_n^2 L \]  
(30)
Set the maximum kinetic energy equal to the maximum potential energy.

\[ 0.203 \rho \omega_n^2 L = 102.4 \frac{EI}{L^3} \]  \hspace{1cm} (31)

\[ \omega_n^2 = 504.0 \frac{EI}{L^4 \rho} \]  \hspace{1cm} (32)

The Rayleigh method thus yields a natural frequency of

\[ \omega_n = \frac{22.4 \sqrt{\frac{EI}{\rho}}}{L^2} \]  \hspace{1cm} (33)

Note that theoretical value from Reference 1 is

\[ \omega_n = \left[ \frac{22.373}{L^2} \right] \sqrt{\frac{EI}{\rho}} \]  \hspace{1cm} (34)

The analysis is repeating using a trigonometric displacement function as shown in Appendix A. Both displacement functions are shown in Figure 1, normalized to an amplitude of one.
Figure 1.

Reference

Trigonometric Displacement Function

A displacement function which satisfies the four geometric boundary conditions is

\[ y(\xi) = 1 - \cos(2\pi\xi) \]  \hfill (A-1)

\[ \frac{d}{d\xi} y(\xi) = 2\pi \sin(2\pi\xi) \]  \hfill (A-2)

\[ \frac{d^2}{d\xi^2} y(\xi) = \left(4\pi^2\right) \cos(2\pi\xi) \]  \hfill (A-3)

Note the amplitude scale factor for equation (1) is arbitrary.

The total potential energy is

\[ P = \frac{EI}{2L^3} \int_0^1 \left(\left(4\pi^2\right) \cos(2\pi\xi)\right)^2 d\xi \]  \hfill (A-4)

\[ P = \left(16\pi^4\right) \frac{EI}{2L^3} \int_0^1 \left(\cos(2\pi\xi)\right)^2 d\xi \]  \hfill (A-5)

\[ P = \left(16\pi^4\right) \frac{EI}{2L^3} \int_0^1 \left[\frac{1}{2} + \frac{1}{2} \cos(4\pi\xi)\right] d\xi \]  \hfill (A-6)

\[ P = \left(16\pi^4\right) \frac{EI}{2L^3} \left[\frac{1}{2} \xi + \frac{1}{8\pi} \sin(4\pi)\right]_0^1 \]  \hfill (A-7)
\[ P = \left(16\pi^4\right) \frac{EI}{2L^3} \left[ \frac{1}{2} \right] \quad \text{(A-8)} \]

\[ P = \frac{4\pi^4 EI}{L^3} \quad \text{(A-9)} \]

The total kinetic energy is

\[ T = \frac{1}{2} \rho \omega_n^2 L \int_0^1 \left[ 1 - \cos(2\pi \xi) \right]^2 d\xi \quad \text{(A-10)} \]

\[ T = \frac{1}{2} \rho \omega_n^2 L \int_0^1 \left[ 1 - 2\cos(2\pi \xi) + \cos^2(2\pi \xi) \right] d\xi \quad \text{(A-11)} \]

\[ T = \frac{1}{2} \rho \omega_n^2 L \int_0^1 \left[ 1 - 2\cos(2\pi \xi) + \frac{1}{2} + \frac{1}{2} \cos(4\pi \xi) \right] d\xi \quad \text{(A-12)} \]

\[ T = \frac{1}{2} \rho \omega_n^2 L \int_0^1 \left[ \frac{3}{2} - 2\cos(2\pi \xi) + \frac{1}{2} \cos(4\pi \xi) \right] d\xi \quad \text{(A-13)} \]

\[ T = \frac{1}{2} \rho \omega_n^2 L \left[ \frac{3\xi}{2} - \frac{1}{\pi} \sin(2\pi \xi) + \frac{1}{8\pi} \sin(4\pi \xi) \right]_0^1 \quad \text{(A-14)} \]

\[ T = \frac{3}{4} \rho \omega_n^2 L \quad \text{(A-15)} \]

Equate the maximum kinetic energy with the maximum potential energy.

\[ \frac{3}{4} \rho \omega_n^2 L = \frac{4\pi^4 EI}{L^3} \quad \text{(A-16)} \]
\[ \omega_n^2 = \frac{16\pi^4}{3} \frac{EI}{L^4\rho} \]  
(A-17)

The Rayleigh method thus yields a natural frequency of

\[ \omega_n = \frac{22.8}{L^2} \sqrt[2]{\frac{EI}{\rho}} \]  
(A-18)

Again, the theoretical value from Reference 1 is

\[ \omega_n = \left[ \frac{22.373}{L^2} \right] \sqrt[2]{\frac{EI}{\rho}} \]  
(A-19)

APPENDIX B

Matlab Script for Polynomial Analysis

```matlab
clear x;
clear P;
clear T;

% P = @(x)((32-192*x+192*x.^2).^2);
P = @(x)((32-192*x+192*x.^2).*2);
QP = quad(P,0,1)

% T=@(x)((16*x.^2-32*x.^3+16*x.^4).^2);
T=@(x)((16*x.^2-32*x.^3+16*x.^4).*2);
QT = quad(T,0,1)

b=QP/QT;
out5 = sprintf( ' QP/QT= %8.2f 
' ,b);
disp(out5)

% a=sqrt(QP/QT);
% disp('');
out5 = sprintf( ' \omega_n = %6.1f [sqrt(EI/\rho)]/L^2 \n',a);
disp(out5)
```

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