THE NATURAL FREQUENCY OF A RECTANGULAR PLATE
WITH FIXED-FIXED-FIXED-FIXED BOUNDARY CONDITIONS
Revision B

By Tom Irvine
Email: tom@vibrationdata.com

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Introduction

The Rayleigh method is used in this tutorial to determine the fundamental bending frequency. A displacement function is assumed which satisfies the geometric boundary conditions. The geometric conditions are the displacement and slope conditions at the boundaries.

The assumed displacement function is substituted into the strain and kinetic energy equations.

The Rayleigh method gives a natural frequency that is an upper limited of the true natural frequency. The method would give the exact natural frequency if the true displacement function were used. The true displacement function is called an eigenfunction.

Consider the rectangular plate in Figure 1. The largest dimension may be either a or b.

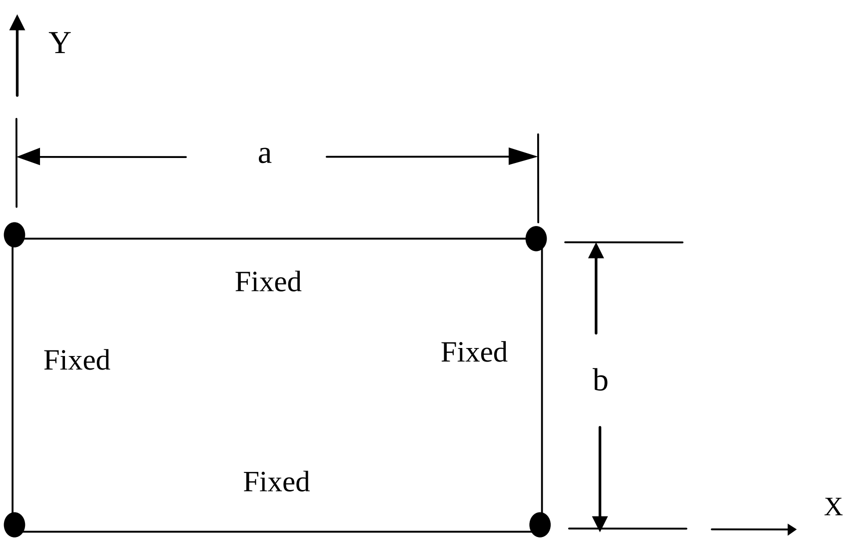

Figure 1.

Let Z represent the out-of-plane displacement. The total strain energy V of the plate is

$$V = \frac{D}{2}\int_0^b\int_0^a \left[\left(\frac{\partial^2 Z}{\partial X^2}\right)^2 + \left(\frac{\partial^2 Z}{\partial Y^2}\right)^2 + 2\mu\left(\frac{\partial^2 Z}{\partial X^2}\right)\left(\frac{\partial^2 Z}{\partial Y^2}\right) + 2(1-\mu)\left(\frac{\partial^2 Z}{\partial X \partial Y}\right)^2\right] dX dY \tag{1}$$

Note that the plate stiffness factor D is given by

$$D = \frac{Eh^3}{12\ (1-\mu^2\ )} \tag{2}$$

where

E	=	elastic modulus
h	=	plate thickness
μ	=	Poisson's ratio

The total kinetic energy T of the plate bending is given by

$$T = \frac{\rho h \Omega^2}{2}\int_0^b\int_0^a Z^2\ dX\,dY \tag{3}$$

where

ρ	=	mass per volume
Ω	=	angular natural frequency

Rayleigh's method can be applied as

$$T_{max} = V_{max} \quad = \text{ total energy of the system} \tag{4}$$

Fixed-Fixed-Fixed-Fixed Plate

Consider the plate in Figure 1.

Seek a displacement function Z(x, y). The geometric boundary conditions are

$$Z(x, y) = 0 \quad \text{at} \quad x=0 \quad \text{and} \quad x=a \tag{5}$$

$$Z(x, y) = 0 \quad \text{at} \quad y=0 \quad \text{and} \quad y=b \tag{6}$$

$$\frac{\partial Z}{\partial x} = 0 \quad \text{at} \quad x=0 \quad \text{and} \quad x=a \tag{7}$$

$$\frac{\partial Z}{\partial y} = 0 \quad \text{at} \quad y=0 \quad \text{and} \quad y=b \tag{8}$$

The candidate displacement function is

$$Z(x, y) = P(x)\,W(y) \tag{9}$$

where

$$P(x) = \left[\cosh(\beta_x x) - \cos(\beta_x x)\right] - \left[\frac{\sinh(\beta L) + \sin(\beta L)}{\cosh(\beta L) - \cos(\beta L)}\right]\left[\sinh(\beta_x x) - \sin(\beta_x x)\right] \tag{10}$$

$$W(y) = \left[\cosh(\beta_y y) - \cos(\beta_y y)\right] - \left[\frac{\sinh(\beta L) + \sin(\beta L)}{\cosh(\beta L) - \cos(\beta L)}\right]\left[\sinh(\beta_y y) - \sin(\beta_y y)\right] \tag{11}$$

$$\beta L = 4.73004 \tag{12}$$

$$\beta_x = 4.73004 / a \tag{13a}$$

$$\beta_y = 4.73004 / b \tag{13b}$$

Note that the candidate function uses two independent beam functions.

The derivatives are

$$\frac{\partial}{\partial x} Z(x, y) = \frac{d}{dx} P(x)\, W(y) \tag{14}$$

$$\frac{\partial^2}{\partial x^2} Z(x, y) = \frac{d^2}{dx^2} P(x)\, W(y) \tag{15}$$

$$\frac{\partial}{\partial y} Z(x, y) = P(x) \frac{d}{dy} W(y) \tag{16}$$

$$\frac{\partial^2}{\partial y^2} Z(x, y) = P(x) \frac{d^2}{dy^2} W(y) \tag{17}$$

$$\frac{dP}{dx} = \beta_x \left\{ \left[\sinh(\beta_x x) + \sin(\beta_x x)\right] - \left[\frac{\sinh(\beta L) + \sin(\beta L)}{\cosh(\beta L) - \cos(\beta L)} \right] \left[\cosh(\beta_x x) - \cos(\beta_x x)\right] \right\} \tag{18}$$

$$\frac{d^2 P}{dx^2} = \beta_x \left\{ \left[\cosh(\beta_x x) + \cos(\beta_x x)\right] - \left[\frac{\sinh(\beta L) + \sin(\beta L)}{\cosh(\beta L) - \cos(\beta L)} \right] \left[\sinh(\beta_x x) + \sin(\beta_x x)\right] \right\} \tag{19}$$

$$\frac{dW}{dy} = \beta_y \left\{ \left[\sinh(\beta_y y) + \sin(\beta_y y)\right] - \left[\frac{\sinh(\beta L) + \sin(\beta L)}{\cosh(\beta L) - \cos(\beta L)} \right] \left[\cosh(\beta_y y) - \cos(\beta_y y)\right] \right\} \tag{20}$$

$$\frac{d^2W}{dy^2} = \beta_y \left\{ \left[\cosh(\beta_y y) + \cos(\beta_y y) \right] - \left[\frac{\sinh(\beta L) + \sin(\beta L)}{\cosh(\beta L) - \cos(\beta L)} \right] \left[\sinh(\beta_y y) + \sin(\beta_y y) \right] \right\} \tag{21}$$

The candidate displacement function satisfies the geometric boundary conditions.

Now equate the total kinetic energy with the total strain energy per Rayleigh's method, equation (4).

This is done numerically via Matlab script: fixed_fixed_fixed_fixed_plate.m

The integrals are converted to series form for this calculation.

<u>Mass-Normalized Mode Shape</u>

The mode shapes are normalized as

$$\rho h \int_0^b \int_0^a \left[Z(x, y) \right]^2 dxdy = 1 \tag{22}$$

The mass-normalized mode shape is

$$Z(x, y) = \frac{1}{\sqrt{\rho abh}} \left\{ \left[\cosh(\beta_x x) - \cos(\beta_x x) \right] - \sigma \left[\sinh(\beta_x x) - \sin(\beta_x x) \right] \right\}$$

$$\bullet \left\{ \left[\cosh(\beta_y y) - \cos(\beta_y y) \right] - \sigma \left[\sinh(\beta_y y) - \sin(\beta_y y) \right] \right\} \tag{23}$$

where

$$\sigma = \left[\frac{\sinh(\beta L) + \sin(\beta L)}{\cosh(\beta L) - \cos(\beta L)} \right] \tag{24}$$

Participation Factor

The participation factor for constant mass density is

$$\Gamma = \rho h \int_0^b \int_0^a Z(x,y)\,dxdy \qquad (25)$$

The numerical result is

$$\Gamma = 0.690\sqrt{\rho abh} \qquad (26)$$

Reference Formula

The following formula taken from Blevin's text can be used as an approximation to check the Rayleigh natural frequency result.

$$f_n \approx \frac{\lambda}{2\pi a^2}\sqrt{\frac{D}{\rho}} \quad , \quad \text{where b is the free edge length} \qquad (27)$$

The λ value is found from the following table.

a/b	λ
0.4	23.65
0.67	27.01
1.0	35.99
1.5	60.77
2.5	147.80

The table can be approximated by

$$\lambda = -0.426\,(a/b)^3 + 27.3\,(a/b)^2 - 16.9\,(a/b) + 26.1 \qquad (28)$$

Example

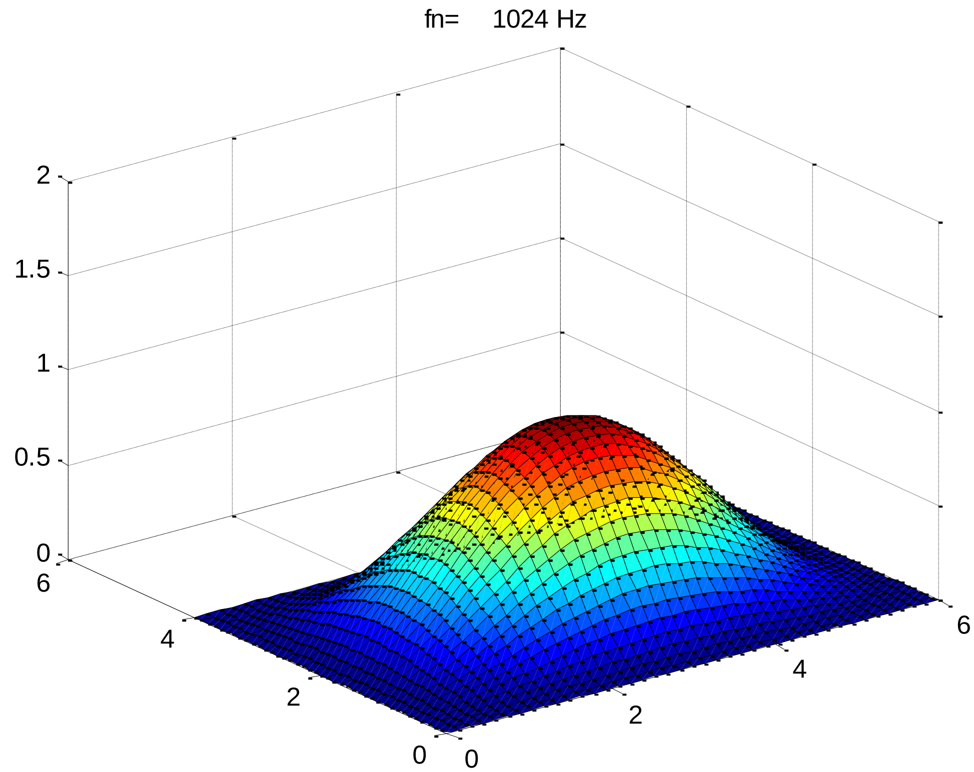

Figure 2.

A fixed-fixed-fixed-fixed aluminum plate has dimensions:

Length	=	6 in
Width	=	4 in
Thickness	=	0.063 in

The elastic modulus is 1.0e+07 lbf/in^2. The mass density is 0.1 lbm/in^3.

The Blevins expected natural frequency is 1016 Hz

The Rayleigh natural frequency is 1024 Hz as calculated using the method derived in this paper. The mode shape is shown in Figure 2.

The Rayleigh method accuracy can be improved using the Rayleigh-Ritz method.

Reference

1. R. Blevins, Formulas for Natural Frequency and Mode Shape, Krieger, Malabar, Florida, 1979. See Table 11-6.