THE FOURIER SERIES OF A RECTANGULAR WAVE Revision A

By Tom Irvine
Email: tomirvine@aol.com

June 13, 2005

A signal may be represented in terms of a Fourier series.

$$f(t) = \frac{1}{2}a_o + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right) \tag{1}$$

The period of the signal is T.

The coefficients are

$$a_o = \left(\frac{2}{T}\right)\int_{-T/2}^{T/2} f(t)dt \tag{2}$$

$$a_n = \left(\frac{2}{T}\right)\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi nt}{T}\right)dt \tag{3}$$

$$b_n = \left(\frac{2}{T}\right)\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi nt}{T}\right)dt \tag{4}$$

Consider a rectangular wave with an infinite number of cycles. The one-cycle segment of the rectangular wave centered about t=0 is

$$f(t) = \begin{cases} -1 & \text{for } -T/2 < t < 0 \\ 0 & \text{for } t = 0 \\ 1 & \text{for } 0 < t < T/2 \end{cases} \tag{5}$$

The coefficients for a rectangular wave are

$$a_n = -\left(\frac{2}{T}\right)\int_{-T/2}^{0}\cos\left(\frac{2\pi nt}{T}\right)dt + \left(\frac{2}{T}\right)\int_{0}^{T/2}\cos\left(\frac{2\pi nt}{T}\right)dt \tag{6}$$

$$a_n = -\frac{1}{\pi n}\sin\left(\frac{2\pi nt}{T}\right)\Bigg|_{-T/2}^{0} + \frac{1}{\pi n}\sin\left(\frac{2\pi nt}{T}\right)\Bigg|_{0}^{T/2} \tag{7}$$

$$a_n = 0 \tag{8}$$

$$b_n = -\left(\frac{2}{T}\right)\int_{-T/2}^{0}\sin\left(\frac{2\pi nt}{T}\right)dt + \left(\frac{2}{T}\right)\int_{0}^{T/2}\sin\left(\frac{2\pi nt}{T}\right)dt \tag{9}$$

$$b_n = \frac{1}{\pi n}\cos\left(\frac{2\pi nt}{T}\right)\Bigg|_{-T/2}^{0} - \frac{1}{\pi n}\cos\left(\frac{2\pi nt}{T}\right)\Bigg|_{0}^{T/2} \tag{10}$$

$$b_n = \frac{1}{\pi n}[1-\cos(-n\pi)] - \frac{1}{\pi n}[\cos(n\pi)-1] \tag{11}$$

$$b_n = \frac{2}{\pi n}[\,1-\cos(n\pi)\,] \tag{12}$$

Thus, the Fourier transform for a rectangular wave is

$$f(t) = 2\sum_{n=1}^{N}\frac{1}{n\pi}[\,1-\cos(n\pi)\,]\sin\left(\frac{2\pi n\,t}{T}\right), \quad \text{where } N \rightarrow \infty \tag{13}$$

Let T=1 seconds. Equation (13) is plotted for several N values in Figure 1. It is plotted for N=10000 in Figure 2.

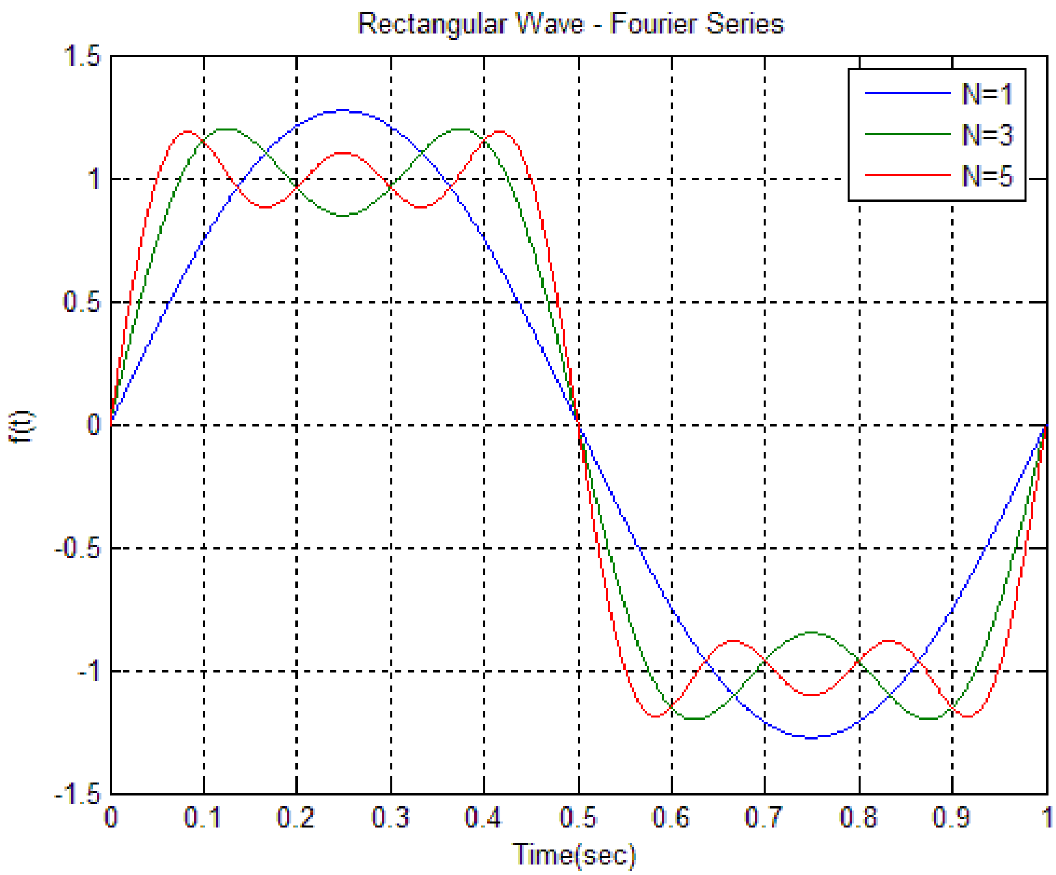

Figure 1.

Note: color plot.

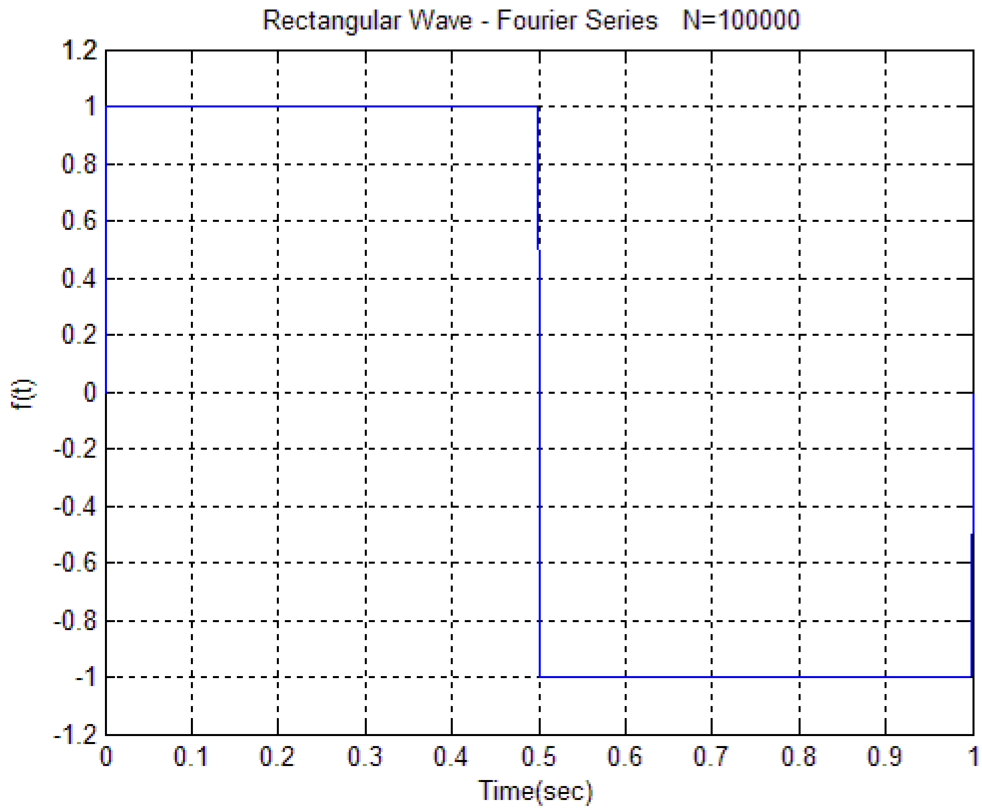
Rectangular Wave - Fourier Series N=100000
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
f(t)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Time(sec)

Figure 2.

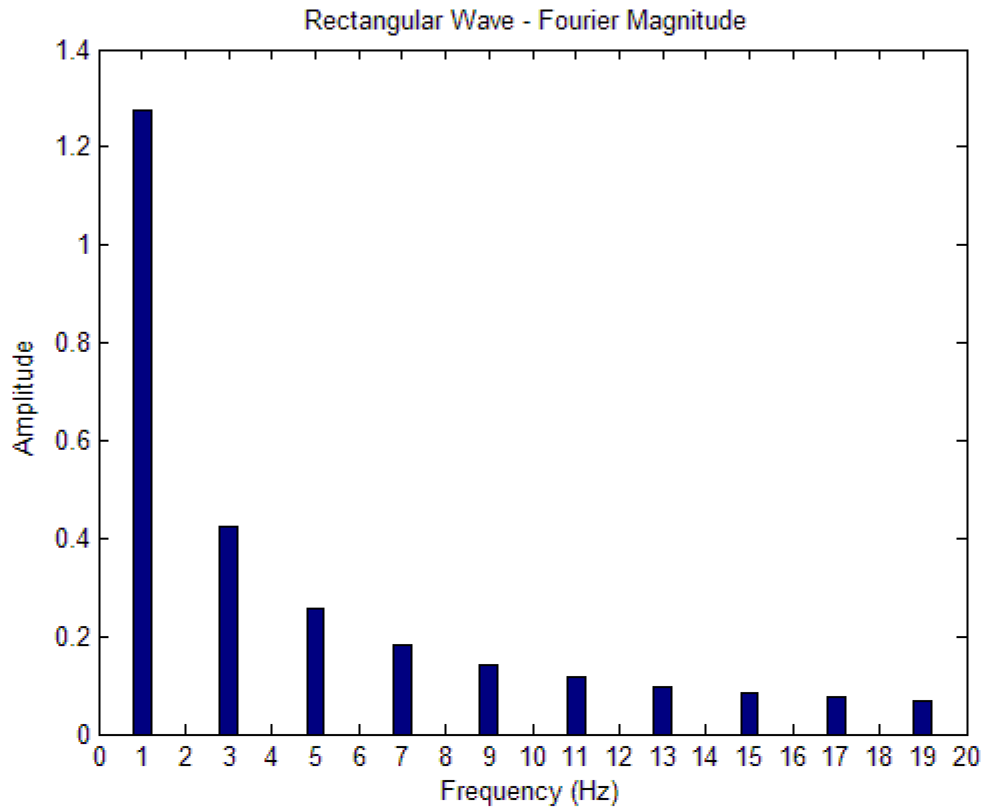

Figure 3.

The first twenty spectral lines are shown. Note that the even components have an amplitude of zero.

APPENDIX A

Matlab Program

```
disp(' ');
disp(' rectangular.m   ver 1.0  June 13, 2005');
disp(' by Tom Irvine');
disp(' ');
disp(' This program calculates the Fourier series of a
rectangular wave. ');
disp(' ');
%
clear all;
%
disp(' Enter frequency (Hz) ');
fn=input(' ');
%
disp(' Enter N limit ');
LIMIT=input(' ');
%
num=4000;
T=1/fn;
dt=T/num;
tpi=2.*pi;
%
for(i=1:(num+1))
        t=(i-1)*dt;
        f(i)=0.;
        TT=tpi*t/T;
        for(m=1:LIMIT)
            n=LIMIT-m+1;
            f(i)=f(i)+(1/n)*(1.-cos(n*pi))*sin(n*TT);
        end
        time(i)=t;
end
f=2.*f/pi;
%
for(n=1:20)
    spectralf(n)=n/T;
    spectrala(n)=(2/(n*pi))*(1.-cos(n*pi));
end
%
figure(1);
plot(time,f);
xlabel('Time(sec)')
```

```
ylabel('f(t)')
out1=sprintf('Rectangular Wave - Fourier Series   N=%d
',LIMIT);
title(out1);
grid on;
set(gca,'MinorGridLineStyle','none','GridLineStyle',':','XS
cale','lin','YScale','lin');
%
disp(' ');
disp(' Plot Fourier magnitude? ');
choice=input('1=yes 2=no  ');
if(choice==1)
    figure(2);
    bar(spectralf,spectrala,0.4);
    title('Rectangular Wave - Fourier Magnitude ');
    xlabel('Frequency (Hz)');
    ylabel('Amplitude ');
end
```