VIBRATION RESPONSE OF A RIGID RECTANGULAR PLATE OR FRAME MOUNTED ON FOUR SPRINGS

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Consider a rigid plate or rigid plane frame mounted via a spring at each corner, as shown in Figure 1. A frame is shown for convenience, but the model can also be applied to a plate. The motion is constrained to translation in the X-axis only.

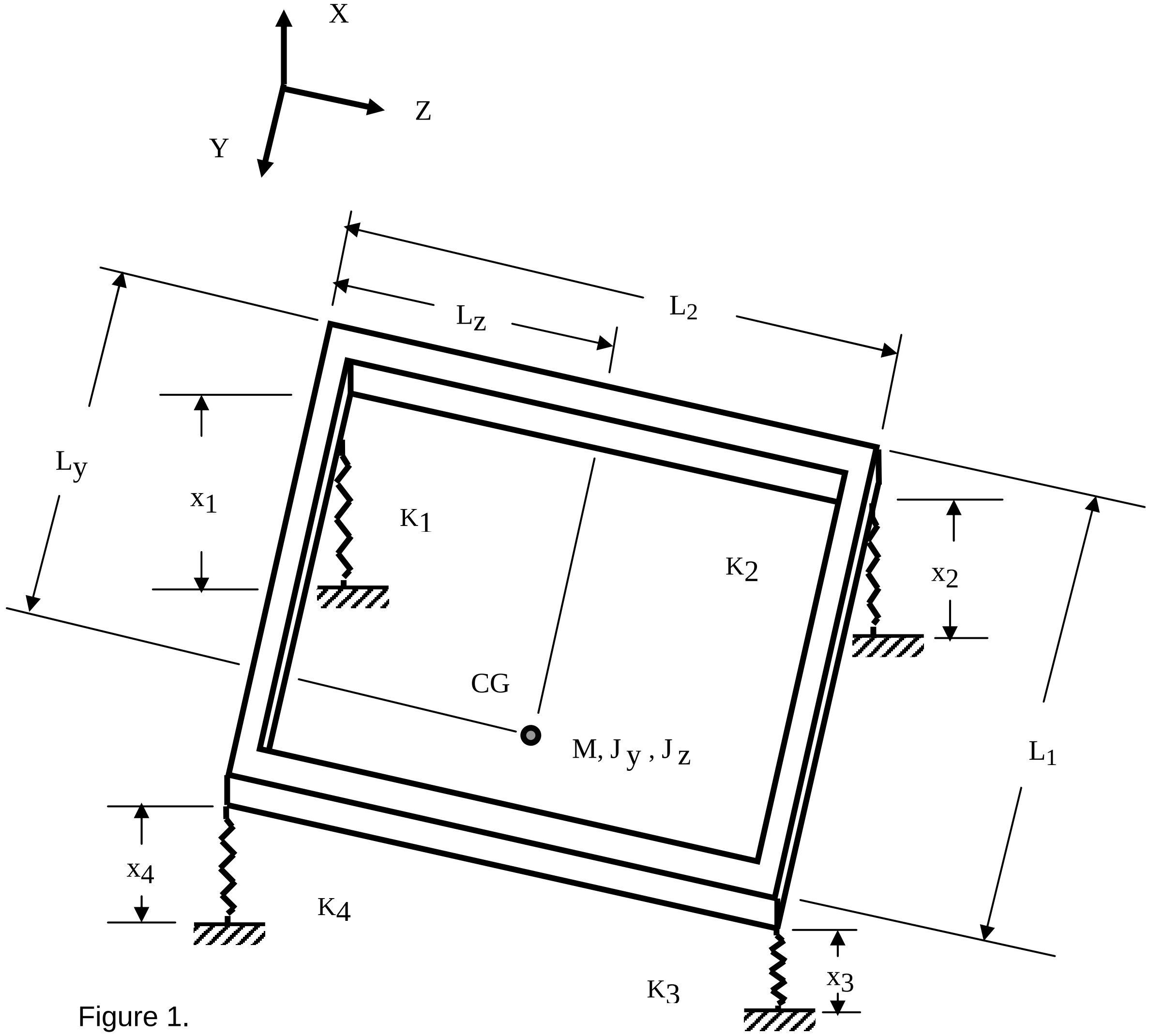


Figure 1.

Determine the natural frequencies and mode shapes via a matrix method.

The total energy of a conservative system is constant. Thus,

$$\frac{d}{dt}\left(KE+PE\right)=0 \tag{1}$$

where

KE = kinetic energy
PE = potential energy

The total potential energy is

$$PE = \frac{1}{2}k_1[x_1]^2+\frac{1}{2}k_2[x_2]^2+\frac{1}{2}k_3[x_3]^2+\frac{1}{2}k_4[x_4]^2 \tag{2}$$

The total kinetic energy is

$$\begin{aligned} KE = &\frac{1}{2}M\left[\dot{x}_1\left(1-\frac{L_z}{L_2}\right)\left(1-\frac{L_y}{L_1}\right)+\dot{x}_2\left(\frac{L_z}{L_2}\right)\left(1-\frac{L_y}{L_1}\right)+\dot{x}_3\left(\frac{L_z}{L_2}\right)\left(\frac{L_y}{L_1}\right)+\dot{x}_4\left(1-\frac{L_z}{L_2}\right)\left(\frac{L_y}{L_1}\right)\right]^2 \\ &+\frac{1}{2}Jy\left[\dot{x}_1\left(-\frac{1}{L_2}\right)\left(1-\frac{L_y}{L_1}\right)+\dot{x}_2\left(\frac{1}{L_2}\right)\left(1-\frac{L_y}{L_1}\right)+\dot{x}_3\left(\frac{1}{L_2}\right)\left(\frac{L_y}{L_1}\right)+\dot{x}_4\left(-\frac{1}{L_2}\right)\left(\frac{L_y}{L_1}\right)\right]^2 \\ &+\frac{1}{2}Jz\left[\dot{x}_1\left(1-\frac{L_z}{L_2}\right)\left(-\frac{1}{L_1}\right)+\dot{x}_2\left(\frac{L_z}{L_2}\right)\left(-\frac{1}{L_1}\right)+\dot{x}_3\left(\frac{L_z}{L_2}\right)\left(\frac{1}{L_1}\right)+\dot{x}_4\left(1-\frac{L_z}{L_2}\right)\left(\frac{1}{L_1}\right)\right]^2 \end{aligned} \tag{3}$$

Let

$$\alpha=\frac{L_y}{L_1} \quad , \quad \alpha'=\frac{1}{L_1} \quad , \quad \lambda=1-\alpha \tag{4}$$

$$\beta=\frac{L_z}{L_2} \quad , \quad \beta'=\frac{1}{L_2} \quad , \quad \sigma=1-\beta \tag{5}$$

$$
\begin{aligned}
&\frac{d}{dt}(KE+PE) \\
&= M[\dot{x}_1(\sigma)(\lambda)+\dot{x}_2(\beta)(\lambda)+\dot{x}_3(\beta)(\alpha)+\dot{x}_4(\sigma)(\alpha)]\ddot{x}_1(\sigma)(\lambda) \\
&+ M[\dot{x}_1(\sigma)(\lambda)+\dot{x}_2(\beta)(\lambda)+\dot{x}_3(\beta)(\alpha)+\dot{x}_4(\sigma)(\alpha)]\ddot{x}_2(\beta)(\lambda) \\
&+ M[\dot{x}_1(\sigma)(\lambda)+\dot{x}_2(\beta)(\lambda)+\dot{x}_3(\beta)(\alpha)+\dot{x}_4(\sigma)(\alpha)]\ddot{x}_3(\beta)(\alpha) \\
&+ M[\dot{x}_1(\sigma)(\lambda)+\dot{x}_2(\beta)(\lambda)+\dot{x}_3(\beta)(\alpha)+\dot{x}_4(\sigma)(\alpha)]\ddot{x}_4(\sigma)(\alpha) \\
&+ Jy[\dot{x}_1(-\beta')(\lambda)+\dot{x}_2(\beta')(\lambda)+\dot{x}_3(\beta')(\alpha)+\dot{x}_4(-\beta')(\alpha)]\ddot{x}_1(-\beta')(\lambda) \\
&+ Jy[\dot{x}_1(-\beta')(\lambda)+\dot{x}_2(\beta')(\lambda)+\dot{x}_3(\beta')(\alpha)+\dot{x}_4(-\beta')(\alpha)]\ddot{x}_2(\beta')(\lambda) \\
&+ Jy[\dot{x}_1(-\beta')(\lambda)+\dot{x}_2(\beta')(\lambda)+\dot{x}_3(\beta')(\alpha)+\dot{x}_4(-\beta')(\alpha)]\ddot{x}_3(\beta')(\alpha) \\
&+ Jy[\dot{x}_1(-\beta')(\lambda)+\dot{x}_2(\beta')(\lambda)+\dot{x}_3(\beta')(\alpha)+\dot{x}_4(-\beta')(\alpha)]\ddot{x}_4(-\beta')(\alpha) \\
&+ Jz[\dot{x}_1(\sigma)(-\alpha')+\dot{x}_2(\beta)(-\alpha')+\dot{x}_3(\beta)(\alpha')+\dot{x}_4(\sigma)(\alpha')]\ddot{x}_1(\sigma)(-\alpha') \\
&+ Jz[\dot{x}_1(\sigma)(-\alpha')+\dot{x}_2(\beta)(-\alpha')+\dot{x}_3(\beta)(\alpha')+\dot{x}_4(\sigma)(\alpha')]\ddot{x}_2(\beta)(-\alpha') \\
&+ Jz[\dot{x}_1(\sigma)(-\alpha')+\dot{x}_2(\beta)(-\alpha')+\dot{x}_3(\beta)(\alpha')+\dot{x}_4(\sigma)(\alpha')]\ddot{x}_3(\beta)(\alpha') \\
&+ Jz[\dot{x}_1(\sigma)(-\alpha')+\dot{x}_2(\beta)(-\alpha')+\dot{x}_3(\beta)(\alpha')+\dot{x}_4(\sigma)(\alpha')]\ddot{x}_4(\sigma)(\alpha') \\
&\qquad + k_1[x_1]\dot{x}_1 + k_2[x_2]\dot{x}_2 + k_3[x_3]\dot{x}_3 + \frac{1}{2}k_4[x_4]\dot{x}_4 \\
&= 0
\end{aligned}
\tag{6}
$$

Consider each of the four degrees-of-freedom individually.

dof 1

$$M[\dot{x}_1(\sigma)(\lambda)]\ddot{x}_1(\sigma)(\lambda) + M[\dot{x}_1(\sigma)(\lambda)]\ddot{x}_2(\beta)(\lambda)$$
$$+ M[\dot{x}_1(\sigma)(\lambda)]\ddot{x}_3(\beta)(\alpha) + M[\dot{x}_1(\sigma)(\lambda)]\ddot{x}_4(\sigma)(\alpha)$$

$$+ Jy[\dot{x}_1(-\beta')(\lambda)]\ddot{x}_1(-\beta')(\lambda) + Jy[\dot{x}_1(-\beta')(\lambda)]\ddot{x}_2(\beta')(\lambda)$$
$$+ Jy[\dot{x}_1(-\beta')(\lambda)]\ddot{x}_3(\beta')(\alpha) + Jy[\dot{x}_1(-\beta')(\lambda)]\ddot{x}_4(-\beta')(\alpha)$$

$$+ Jz[\dot{x}_1(\sigma)(-\alpha')]\ddot{x}_1(\sigma)(-\alpha') + Jz[\dot{x}_1(\sigma)(-\alpha')]\ddot{x}_2(\beta)(-\alpha')$$
$$+ Jz[\dot{x}_1(\sigma)(-\alpha')]\ddot{x}_3(\beta)(\alpha') + Jz[\dot{x}_1(\sigma)(-\alpha')]\ddot{x}_4(\sigma)(\alpha')$$

$$+ k_1[x_1]\dot{x}_1 \quad = \quad 0$$

(7)

$$\dot{x}_1 M\left\{\left(\sigma^2\lambda^2\right)\ddot{x}_1 + \left(\sigma\beta\lambda^2\right)\ddot{x}_2 + (\alpha\beta\sigma\lambda)\ddot{x}_3 + \left(\alpha\lambda\sigma^2\right)\ddot{x}_4\right\}$$

$$+ \dot{x}_1 Jy\left\{\left[-(\beta')^2\lambda^2\right]\ddot{x}_1 + \left[-(\beta')^2\lambda^2\right]\ddot{x}_2 + \left[-(\beta')^2(\alpha\lambda)\right]\ddot{x}_3 + \left[(\beta')^2(\alpha\lambda)\right]\ddot{x}_4\right\}$$

$$+ \dot{x}_1 Jz\left\{\left[\sigma^2(\alpha')^2\right]\ddot{x}_1 + \left[\sigma\beta(\alpha')^2\right]\ddot{x}_2 + \left[-\sigma\beta(\alpha')^2\right]\ddot{x}_3 + \left[-\sigma^2(\alpha')^2\right]\ddot{x}_4\right\}$$

$$+ \dot{x}_1 k_1[x_1] \quad = \quad 0$$

(8)

$$M\left\{\left(\sigma^2\lambda^2\right)\ddot{x}_1 + \left(\sigma\beta\lambda^2\right)\ddot{x}_2 + (\alpha\beta\sigma\lambda)\ddot{x}_3 + \left(\alpha\lambda\sigma^2\right)\ddot{x}_4\right\}$$
$$+ Jy\left\{\left[-(\beta')^2\lambda^2\right]\ddot{x}_1 + \left[-(\beta')^2\lambda^2\right]\ddot{x}_2 + \left[-(\beta')^2(\alpha\lambda)\right]\ddot{x}_3 + \left[(\beta')^2(\alpha\lambda)\right]\ddot{x}_4\right\}$$
$$+ Jz\left\{\left[\sigma^2(\alpha')^2\right]\ddot{x}_1 + \left[\sigma\beta(\alpha')^2\right]\ddot{x}_2 + \left[-\sigma\beta(\alpha')^2\right]\ddot{x}_3 + \left[-\sigma^2(\alpha')^2\right]\ddot{x}_4\right\}$$

$$+ k_1[x_1] \quad = \quad 0$$

(9)

$$
\begin{aligned}
&\left\{M\left(\sigma^2\lambda^2\right)+Jy\left[-(\beta')^2\lambda^2\right]+Jz\left[\sigma^2(\alpha')^2\right]\right\}\ddot{x}_1\\
&+\left\{M\left(\sigma\beta\lambda^2\right)+Jy\left[-(\beta')^2\lambda^2\right]+Jz\left[\sigma\beta(\alpha')^2\right]\right\}\ddot{x}_2\\
&+\left\{M(\alpha\beta\sigma\lambda)+Jy\left[-(\beta')^2(\alpha\lambda)\right]+Jz\left[-\sigma\beta(\alpha')^2\right]\right\}\ddot{x}_3\\
&+\left\{M\left(\alpha\lambda\sigma^2\right)+Jy\left[(\beta')^2(\alpha\lambda)\right]+Jz\left[-\sigma^2(\alpha')^2\right]\right\}\ddot{x}_4\\
&+k_1[x_1] \quad = \quad 0
\end{aligned}
\tag{10}
$$

<u>dof 2</u>

$$
\begin{aligned}
&= M[\dot{x}_2(\beta)(\lambda)]\ddot{x}_1(\sigma)(\lambda)+M[\dot{x}_2(\beta)(\lambda)]\ddot{x}_2(\beta)(\lambda)\\
&+M[\dot{x}_2(\beta)(\lambda)]\ddot{x}_3(\beta)(\alpha)+M[\dot{x}_2(\beta)(\lambda)]\ddot{x}_4(\sigma)(\alpha)\\
&+Jy[\dot{x}_2(\beta')(\lambda)]\ddot{x}_1(-\beta')(\lambda)+Jy[\dot{x}_2(\beta')(\lambda)]\ddot{x}_2(\beta')(\lambda)\\
&+Jy[\dot{x}_2(\beta')(\lambda)]\ddot{x}_3(\beta')(\alpha)+Jy[\dot{x}_2(\beta')(\lambda)]\ddot{x}_4(-\beta')(\alpha)\\
&+Jz[\dot{x}_2(\beta)(-\alpha')]\ddot{x}_1(\sigma)(-\alpha')+Jz[\dot{x}_2(\beta)(-\alpha')]\ddot{x}_2(\beta)(-\alpha')\\
&+Jz[\dot{x}_2(\beta)(-\alpha')]\ddot{x}_3(\beta)(\alpha')+Jz[\dot{x}_2(\beta)(-\alpha')]\ddot{x}_4(\sigma)(\alpha')\\
&\qquad +k_2[x_2]\dot{x}_2
\end{aligned}
\tag{11}
$$

$$\dot{x}_2 M\left\{\left[\sigma\beta\lambda^2\right]\ddot{x}_1 + \left[\beta^2\lambda^2\right]\ddot{x}_2 + \left[\alpha\lambda\beta^2\right]\ddot{x}_3 + [\alpha\beta\sigma\lambda]\ddot{x}_4\right\}$$

$$+ \dot{x}_2 Jy\left\{\left[-(\beta')^2\lambda^2\right]\ddot{x}_1 + \left[(\beta')^2\lambda^2\right]\ddot{x}_2 + \left[(\beta')^2\alpha\lambda\right]\ddot{x}_3 + \left[-\alpha\lambda(\beta')^2\right]\ddot{x}_4\right\}$$

$$+ \dot{x}_2 Jz\left\{\left[\sigma\beta(\alpha')^2\right]\ddot{x}_1 + \left[\beta^2(-\alpha')^2\right]\ddot{x}_2 + \left[-\beta^2(\alpha')^2\right]\ddot{x}_3 + \left[-\sigma\beta(\alpha')^2\right]\ddot{x}_4\right\}$$

$$+ k_2[x_2]\dot{x}_2 = 0 \tag{12}$$

$$M\left\{\left[\sigma\beta\lambda^2\right]\ddot{x}_1 + \left[\beta^2\lambda^2\right]\ddot{x}_2 + \left[\alpha\lambda\beta^2\right]\ddot{x}_3 + [\alpha\beta\sigma\lambda]\ddot{x}_4\right\}$$

$$+ Jy\left\{\left[-(\beta')^2\lambda^2\right]\ddot{x}_1 + \left[(\beta')^2\lambda^2\right]\ddot{x}_2 + \left[(\beta')^2\alpha\lambda\right]\ddot{x}_3 + \left[-\alpha\lambda(\beta')^2\right]\ddot{x}_4\right\}$$

$$+ Jz\left\{\left[\sigma\beta(\alpha')^2\right]\ddot{x}_1 + \left[\beta^2(-\alpha')^2\right]\ddot{x}_2 + \left[-\beta^2(\alpha')^2\right]\ddot{x}_3 + \left[-\sigma\beta(\alpha')^2\right]\ddot{x}_4\right\}$$

$$+ k_2[x_2] = 0 \tag{13}$$

$$M\left\{\left[\sigma\beta\lambda^2\right]\ddot{x}_1+\left[\beta^2\lambda^2\right]\ddot{x}_2+\left[\alpha\lambda\beta^2\right]\ddot{x}_3+[\alpha\beta\sigma\lambda]\ddot{x}_4\right\}$$

$$+Jy\left\{\left[-(\beta')^2\lambda^2\right]\ddot{x}_1+\left[(\beta')^2\lambda^2\right]\ddot{x}_2+\left[(\beta')^2\alpha\lambda\right]\ddot{x}_3+\left[-\alpha\lambda(\beta')^2\right]\ddot{x}_4\right\}$$

$$+Jz\left\{\left[\sigma\beta(\alpha')^2\right]\ddot{x}_1+\left[\beta^2(-\alpha')^2\right]\ddot{x}_2+\left[-\beta^2(\alpha')^2\right]\ddot{x}_3+\left[-\sigma\beta(\alpha')^2\right]\ddot{x}_4\right\}$$

$$+k_2[x_2]=0 \tag{14}$$

<u>dof 3</u>

$$=M[\dot{x}_3(\beta)(\alpha)]\ddot{x}_1(\sigma)(\lambda)+M[\dot{x}_3(\beta)(\alpha)]\ddot{x}_2(\beta)(\lambda)$$
$$+M[\dot{x}_3(\beta)(\alpha)]\ddot{x}_3(\beta)(\alpha)+M[\dot{x}_3(\beta)(\alpha)]\ddot{x}_4(\sigma)(\alpha)$$

$$+Jy[\dot{x}_3(\beta')(\alpha)]\ddot{x}_1(-\beta')(\lambda)+Jy[\dot{x}_3(\beta')(\alpha)]\ddot{x}_2(\beta')(\lambda)$$
$$+Jy[\dot{x}_3(\beta')(\alpha)]\ddot{x}_3(\beta')(\alpha)+Jy[\dot{x}_3(\beta')(\alpha)]\ddot{x}_4(-\beta')(\alpha)$$

$$+Jz[\dot{x}_3(\beta)(\alpha')]\ddot{x}_1(\sigma)(-\alpha')+Jz[\dot{x}_3(\beta)(\alpha')]\ddot{x}_2(\beta)(-\alpha')$$
$$+Jz[\dot{x}_3(\beta)(\alpha')]\ddot{x}_3(\beta)(\alpha')+Jz[\dot{x}_3(\beta)(\alpha')]\ddot{x}_4(\sigma)(\alpha')$$

$$+k_3[x_3]\dot{x}_3 \tag{15}$$

$$\dot{x}_3M\left\{[\alpha\beta\sigma\lambda]\ddot{x}_1+\left[\alpha\beta^2\lambda\right]\ddot{x}_2+\left[\alpha^2\beta^2\right]\ddot{x}_3+\left[\alpha^2\sigma\beta\right]\ddot{x}_4\right\}$$
$$+\dot{x}_3Jy\left\{\left[-\alpha\lambda(\beta')^2\right]\ddot{x}_1+\left[\alpha\lambda(\beta')^2\right]\ddot{x}_2+\left[(\beta')^2(\alpha)^2\right]\ddot{x}_3+\left[-\alpha^2(\beta')^2\right]\ddot{x}_4\right\}$$
$$+\dot{x}_3Jz\left\{\left[-\beta\sigma(\alpha')^2\right]\ddot{x}_1+\left[-\beta^2(\alpha')^2\right]\ddot{x}_2+\left[\beta^2(\alpha')^2\right]\ddot{x}_3+\left[\beta\sigma(\alpha')^2\right]\ddot{x}_4\right\}$$

$$+k_3[x_3]\dot{x}_3 \quad = \quad 0 \tag{16}$$

$$M\left\{ [\alpha\beta\sigma\lambda]\ddot{x}_1 + \left[\alpha\beta^2\lambda\right]\ddot{x}_2 + \left[\alpha^2\beta^2\right]\ddot{x}_3 + \left[\alpha^2\sigma\beta\right]\ddot{x}_4 \right\}$$

$$+ Jy\left\{\left[-\alpha\lambda(\beta')^2\right]\ddot{x}_1 + \left[\alpha\lambda(\beta')^2\right]\ddot{x}_2 + \left[(\beta')^2\alpha^2\right]\ddot{x}_3 + \left[-\alpha^2(\beta')^2\right]\ddot{x}_4\right\}$$

$$+ \dot{x}_3 Jz\left\{\left[-\beta\sigma(\alpha')^2\right]\ddot{x}_1 + \left[-\beta^2(\alpha')^2\right]\ddot{x}_2 + \left[\beta^2(\alpha')^2\right]\ddot{x}_3 + \left[\beta\sigma(\alpha')^2\right]\ddot{x}_4\right\}$$

$$+ k_3[x_3] \quad = \quad 0 \tag{17}$$

<u>dof 4</u>

$$\begin{aligned}
&= M[\dot{x}_4(\sigma)(\alpha)]\ddot{x}_1(\sigma)(\lambda) + M[\dot{x}_4(\sigma)(\alpha)]\ddot{x}_2(\beta)(\lambda) \\
&+ M[\dot{x}_4(\sigma)(\alpha)]\ddot{x}_3(\beta)(\alpha) + M[\dot{x}_4(\sigma)(\alpha)]\ddot{x}_4(\sigma)(\alpha) \\
&+ Jy[\dot{x}_4(-\beta')(\alpha)]\ddot{x}_1(-\beta')(\lambda) + Jy[\dot{x}_4(-\beta')(\alpha)]\ddot{x}_2(\beta')(\lambda) \\
&+ Jy[\dot{x}_4(-\beta')(\alpha)]\ddot{x}_3(\beta')(\alpha) + Jy[\dot{x}_4(-\beta')(\alpha)]\ddot{x}_4(-\beta')(\alpha) \\
&+ Jz[\dot{x}_4(\sigma)(\alpha')]\ddot{x}_1(\sigma)(-\alpha') + Jz[\dot{x}_4(\sigma)(\alpha')]\ddot{x}_2(\beta)(-\alpha') \\
&+ Jz[\dot{x}_4(\sigma)(\alpha')]\ddot{x}_3(\beta)(\alpha') + Jz[\dot{x}_4(\sigma)(\alpha')]\ddot{x}_4(\sigma)(\alpha') \\
&\qquad + \frac{1}{2}k_4[x_4]\dot{x}_4
\end{aligned} \tag{18}$$

$$\dot{x}_4 M\left\{\left[\alpha\lambda\left(\sigma^2\right)\right]\ddot{x}_1 + [\sigma\alpha\beta\lambda]\ddot{x}_2 + \left[\sigma\beta\left(\alpha^2\right)\right]\ddot{x}_3 + \left[\alpha^2\sigma^2\right]\ddot{x}_4\right\}$$

$$+ \dot{x}_4 Jy\left\{\left[\alpha\lambda(\beta')^2\right]\ddot{x}_1 + \left[-\alpha\lambda(\beta')^2\right]\ddot{x}_2 + J\left[-(\beta')^2(\alpha)^2\right]\ddot{x}_3 + \left[\alpha^2(\beta')^2\right]\right\}$$

$$+ \dot{x}_4 Jz\left\{-\left[\sigma^2(\alpha')^2\right]\ddot{x}_1 + \left[-\sigma\beta(\alpha')^2\right]\ddot{x}_2 + \left[\sigma\beta(\alpha')^2\right]\ddot{x}_3 + \left[\sigma^2(\alpha')^2\right]\ddot{x}_4\right\}$$

$$+\frac{1}{2}k_4[x_4]\dot{x}_4 = 0 \tag{19}$$

$$M\left\{\left[\alpha\lambda\left(\sigma^2\right)\right]\ddot{x}_1 + [\sigma\alpha\beta\lambda]\ddot{x}_2 + \left[\sigma\beta\left(\alpha^2\right)\right]\ddot{x}_3 + \left[\alpha^2\sigma^2\right]\ddot{x}_4\right\}$$

$$+ Jy\left\{\left[\alpha\lambda(\beta')^2\right]\ddot{x}_1 + \left[-\alpha\lambda(\beta')^2\right]\ddot{x}_2 + J\left[-(\beta')^2(\alpha)^2\right]\ddot{x}_3 + \left[\alpha^2(\beta')^2\right]\right\}$$

$$+ Jz\left\{-\left[\sigma^2(\alpha')^2\right]\ddot{x}_1 + \left[-\sigma\beta(\alpha')^2\right]\ddot{x}_2 + \left[\sigma\beta(\alpha')^2\right]\ddot{x}_3 + \left[\sigma^2(\alpha')^2\right]\ddot{x}_4\right\}$$

$$+\frac{1}{2}k_4[x_4] = 0 \tag{20}$$

The global mass matrix MG is

$$MG = M\begin{bmatrix} \sigma^2\lambda^2 & \sigma\beta\lambda^2 & \alpha\beta\sigma\lambda & \alpha\lambda\sigma^2 \\ \sigma\beta\lambda^2 & \beta^2\lambda^2 & \alpha\beta^2\lambda & \alpha\beta\sigma\lambda \\ \alpha\beta\sigma\lambda & \alpha\beta^2\lambda & \alpha^2\beta^2 & \alpha^2\beta\sigma \\ \alpha\lambda\sigma^2 & \alpha\beta\sigma\lambda & \alpha^2\beta\sigma & \alpha^2\sigma^2 \end{bmatrix}$$

$$+ Jy(\beta')^2\begin{bmatrix} \lambda^2 & -\lambda^2 & -\alpha\lambda & \alpha\lambda \\ -\lambda^2 & \lambda^2 & \alpha\lambda & -\alpha\lambda \\ -\alpha\lambda & \alpha\lambda & \alpha^2 & -\alpha^2 \\ \alpha\lambda & -\alpha\lambda & -\alpha^2 & \alpha^2 \end{bmatrix}$$

$$+ Jz(\alpha')^2\begin{bmatrix} \sigma^2 & \sigma\beta & -\sigma\beta & -\sigma^2 \\ \sigma\beta & \beta^2 & -\beta^2 & -\sigma\beta \\ -\sigma\beta & -\beta^2 & \beta^2 & \sigma\beta \\ -\sigma^2 & -\sigma\beta & \sigma\beta & \sigma^2 \end{bmatrix} \qquad (21)$$

The global stiffness matrix KG is

$$KG = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix} \qquad (22)$$

The natural frequencies are found by solving the corresponding generalized eigenvalue problem.

An example is given in Appendix A.

APPENDIX A

Example

Consider a square plate (L x L) with the CG in the middle. All springs have the same stiffness k.

Assume a constant mass which is independent of the geometry. The length L is arbitrary because it will cancel out in the two inertia matrices.

Let

$$M = 1 \text{ lbm} = 0.00259 \text{ lbf sec^2/in} \tag{A-1}$$

$$K = 250 \text{ lbf/in} \tag{A-2}$$

Note that Jx = Jy = J for this example.

$$J \approx \frac{1}{12} M \left(L^2 \right) \tag{A-3}$$

$$J = 0.0833 \text{ lbm in^2} = 0.000216 \text{ lbf sec^2 in} \tag{A-4}$$

The global mass matrix is

$$MG = \frac{1}{16} M \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \frac{1}{4L^2} J \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} + \frac{1}{4L^2} J \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \tag{A-5}$$

$$MG = \frac{1}{16} M \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \frac{1}{4L^2} J \begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix} \tag{A-6}$$

$$MG = \frac{1}{16}M\begin{bmatrix}1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\end{bmatrix} + \frac{1}{48}M\begin{bmatrix}2 & 0 & -2 & 0\\0 & 2 & 0 & -2\\-2 & 0 & 2 & 0\\0 & -2 & 0 & 2\end{bmatrix} \tag{A-7}$$

The global stiffness matrix is

$$KG = K\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \tag{A-8}$$

The calculation is completed via a Matlab code.

The final, global mass and stiffness matrices are

```
MG =

  1.0e-003 *

    0.2700    0.1619    0.0538    0.1619
    0.1619    0.2700    0.1619    0.0538
    0.0538    0.1619    0.2700    0.1619
    0.1619    0.0538    0.1619    0.2700


KG =

   250     0     0     0
     0   250     0     0
     0     0   250     0
     0     0     0   250
```

Note that the mass unit is lbf sec^2/in. The stiffness unit is lbf/in.

The eigenvalues are

```
lambda =

  1.0e+006 *

    0.3860    1.1562    1.1562       Inf


  Natural Frequencies =

      98.88 Hz
      171.1 Hz
      171.1 Hz
        Inf Hz
```

The first three mode shapes in column format are

```
  -19.6469  -48.0864   -0.2209
  -19.6469   -0.0000   48.0859
  -19.6469   48.0864    0.2209
  -19.6469    0.0000  -48.0859
```

The fourth mode is

```
  1.0e+009 *

    4.2950
   -4.2950
    4.2950
   -4.2950
```

The fourth mode shape is fictitious. It implies that the rigid plate undergoes elastic deformation. The fourth natural frequency and its corresponding mode shape should be disregarded.

Hand Calculation Check

Translation

$$fn = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

The total stiffness is 4 x 250 lbf/in

$$fn = \frac{1}{2\pi}\sqrt{\frac{1000 \text{ lbf / in}}{0.00259 \text{ lbf sec^2/in}}}$$

fn = 98.9 Hz

The frequency agrees with the first natural frequency of the Matlab output.

Rotation

Assume that the length is 1 inch for simplicity. This natural frequency is effectively independent of this value, however, since the length cancels out from the numerator and denominator.

$$fn = \frac{1}{2\pi}\sqrt{\frac{k_r}{J}}$$

k_r = 4 x 250 lbf/in x (0.5 inch)^2 = 250 lbf in

$$J = \frac{1}{12} M \left(L^2 \right)$$

$$J = \frac{1}{12}\left(1 \text{ lbm} \right)\left(1 \text{ inch} \right)^2 = 0.0833 \text{ lbm in}^2$$

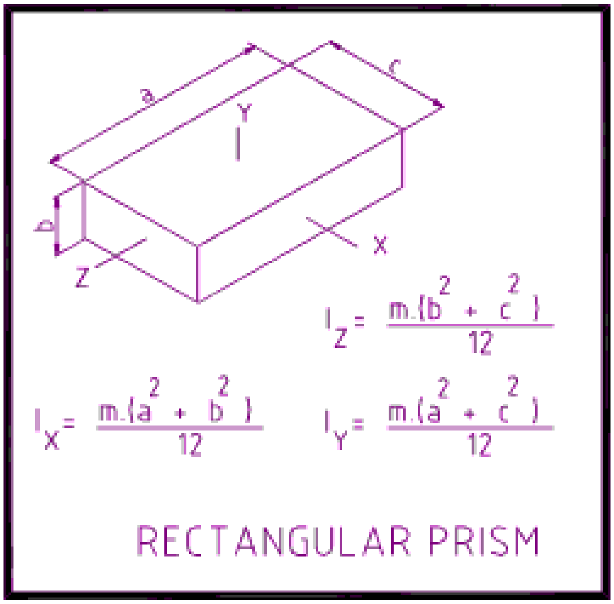
J = 0.000216 lbf sec^2 in

$$fn = \frac{1}{2\pi}\sqrt{\frac{250\ \text{lbf in}}{0.000216\ \text{lbf sec^2 in}}} = 171.3\ \text{Hz}$$

This value agrees with second and third natural frequencies from the Matlab output.

APPENDIX B

Mass Moment of Inertia



http://www.roymech.co.uk/Useful_Tables/Form/Dynamics_inertia.html

$$J \approx \frac{1}{12} M \left(L^2 \right) \quad \text{for a thin plate} \tag{B-1}$$