VIBRATION RESPONSE OF A RIGID RECTANGULAR PLATE OR FRAME SUBJECTED TO MULTIPLE BASE INPUTS Revision A

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Consider a rigid plate or rigid plane frame mounted via a spring at each corner, as shown in Figure 1.

A frame is shown for convenience, but the model can also be applied to a plate. The motion is constrained to translation in the X-axis only.

The mass and stiffness matrices for the generalized eigenvalue problem are shown in Appendix A, as formulated in Reference 1. The natural frequencies are found by solving the corresponding generalized eigenvalue problem.

Now consider that the plate is base-driven by independent acceleration time histories, one at each corner, as shown in Figure 1.

Determine the acceleration response at each corner of the plate and at the CG. Use a modal transient solution.

Assume that there is modal damping. But assume that there are no dashpots connecting the plate to the base.

The potential energy is

$$PE = \frac{1}{2}k_1(x_1 - y_1)^2 + \frac{1}{2}k_2(x_2 - y_2)^2 + \frac{1}{2}k_3(x_3 - y_3)^2 + \frac{1}{2}k_4(x_4 - y_4)^2 \tag{1}$$

$$\frac{d}{dt}PE = k_1(x_1 - y_1)\dot{x}_1 + k_2(x_2 - y_2)\dot{x}_2 + k_3(x_3 - y_3)\dot{x}_3 + k_4(x_4 - y_4)\dot{x}_4 \tag{2}$$

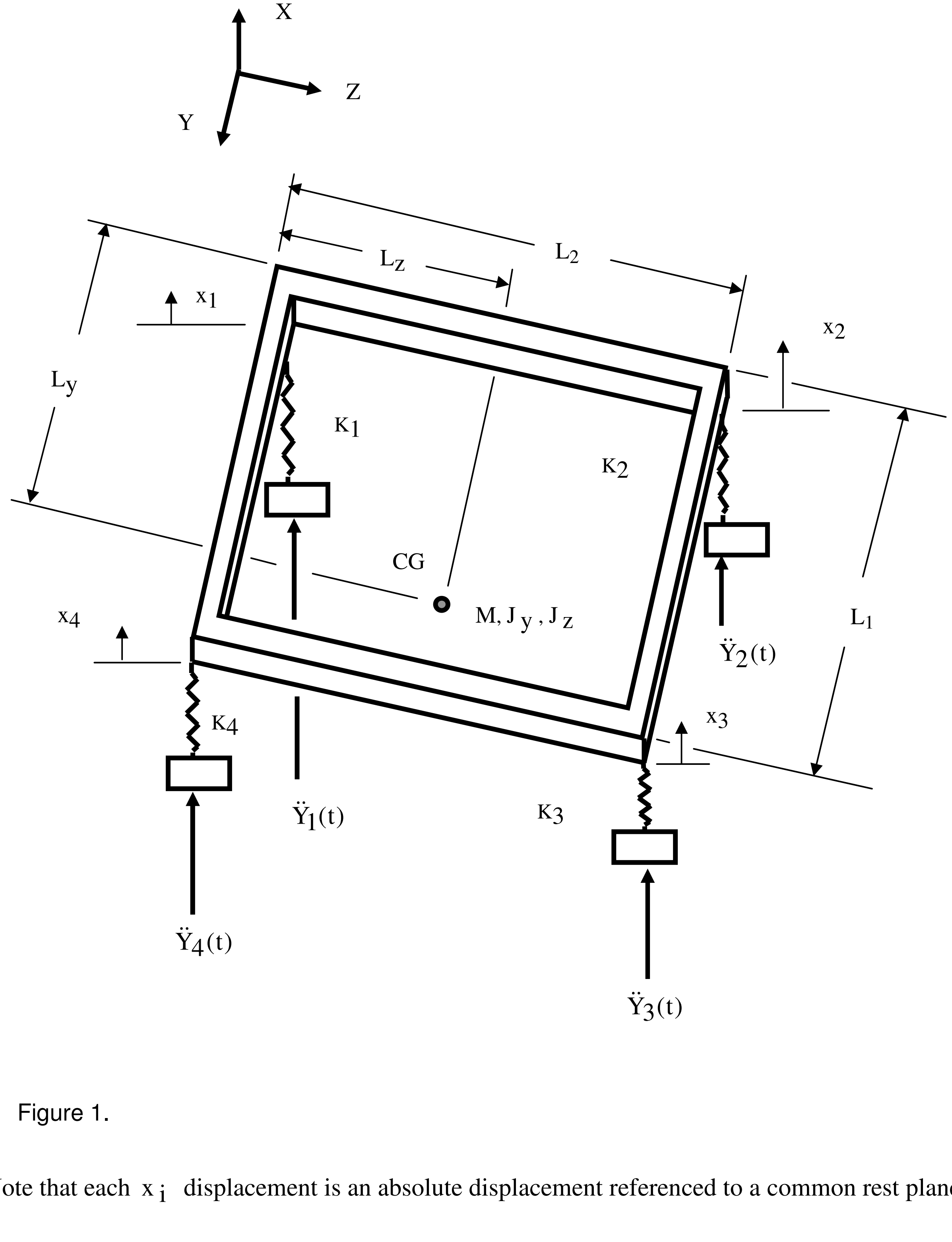


Figure 1.

Note that each x_i displacement is an absolute displacement referenced to a common rest plane.

Equation (2) generates the global stiffness matrix already shown in Appendix A. It also generates the following matrix KG_b which will be inserted to the right-hand-side of the pending non-homogeneous equation. This matrix effectively represents the work done by the base excitation on the springs, when multiplied by the base displacement vector shown in equation (4).

$$KG_b = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix} \tag{3}$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \tag{4}$$

The next step would normally be to develop relative displacement terms to simplify the base input matrices so that the base accelerations can be input directly. This simplification does not appear to be suitable for the case of multiple inputs, however.

The non-homogeneous equation is

$$MG \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + CG \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} + KG \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = KG_b \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \tag{5}$$

The next step is decouple equation (5) using the eigenvectors which are obtained from the undamped, homogeneous equation.

A mass-normalized eigenvector matrix $\hat{Q}$ can be obtained such that the following orthogonality relations are obtained.

$$\hat{Q}^T \, MG \, \hat{Q} = I \tag{6}$$

and

$$\hat{Q}^T \, KG \, \hat{Q} = \Omega \tag{7}$$

where

superscript T represents transpose

I is the identity matrix

Ω is a diagonal matrix of eigenvalues

Note that

$$\hat{Q} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{12} & \hat{q}_{13} & \hat{q}_{14} \\ \hat{q}_{21} & \hat{q}_{22} & \hat{q}_{23} & \hat{q}_{24} \\ \hat{q}_{31} & \hat{q}_{32} & \hat{q}_{33} & \hat{q}_{34} \\ \hat{q}_{41} & \hat{q}_{42} & \hat{q}_{43} & \hat{q}_{44} \end{bmatrix} \tag{8}$$

$$\hat{Q}^T = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} & \hat{q}_{31} & \hat{q}_{41} \\ \hat{q}_{12} & \hat{q}_{22} & \hat{q}_{32} & \hat{q}_{42} \\ \hat{q}_{13} & \hat{q}_{23} & \hat{q}_{33} & \hat{q}_{43} \\ \hat{q}_{14} & \hat{q}_{24} & \hat{q}_{34} & \hat{q}_{44} \end{bmatrix} \tag{9}$$

Rigorous proof of the orthogonality relationships is beyond the scope of this tutorial. Further discussion is given in References 5 and 6.

Nevertheless, the orthogonality relationships are demonstrated by an example in this tutorial.

Now define a modal coordinate $\eta(t)$ such that

$$\bar{x} = \hat{Q} \; \bar{\eta} \tag{10}$$

By substitution.

$$MG\,\hat{Q}\begin{bmatrix}\ddot{\eta}_1\\ \ddot{\eta}_2\\ \ddot{\eta}_3\\ \ddot{\eta}_4\end{bmatrix} + CG\,\hat{Q}\begin{bmatrix}\dot{\eta}_1\\ \dot{\eta}_2\\ \dot{\eta}_3\\ \dot{\eta}_4\end{bmatrix} + KG\,\hat{Q}\begin{bmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\end{bmatrix} = KG_b\begin{bmatrix}y_1\\ y_2\\ y_3\\ y_4\end{bmatrix} \tag{11}$$

Premultiply by the transpose of the normalized eigenvector matrix.

$$\hat{Q}^T\,MG\,\hat{Q}\begin{bmatrix}\ddot{\eta}_1\\ \ddot{\eta}_2\\ \ddot{\eta}_3\\ \ddot{\eta}_4\end{bmatrix} + \hat{Q}^T\,CG\,\hat{Q}\begin{bmatrix}\dot{\eta}_1\\ \dot{\eta}_2\\ \dot{\eta}_3\\ \dot{\eta}_4\end{bmatrix} + \hat{Q}^T\,KG\,\hat{Q}\begin{bmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\end{bmatrix} = \hat{Q}^T\,KG_b\begin{bmatrix}y_1\\ y_2\\ y_3\\ y_4\end{bmatrix} \tag{12}$$

The orthogonality relationships yield

$$I\begin{bmatrix}\ddot{\eta}_1\\ \ddot{\eta}_2\\ \ddot{\eta}_3\\ \ddot{\eta}_4\end{bmatrix} + \hat{C}\begin{bmatrix}\dot{\eta}_1\\ \dot{\eta}_2\\ \dot{\eta}_3\\ \dot{\eta}_4\end{bmatrix} + \Omega\begin{bmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\end{bmatrix} = \hat{Q}^T\,KG_b\begin{bmatrix}y_1\\ y_2\\ y_3\\ y_4\end{bmatrix} \tag{13}$$

The modal damping matrix $\hat{C}$ is substituted for $\hat{Q}^T\,CG\,\hat{Q}$.

$$\hat{C} = \begin{bmatrix}2\xi_1\,\omega_1 & 0 & 0 & 0\\ 0 & 2\xi_2\,\omega_2 & 0 & 0\\ 0 & 0 & 2\xi_3\,\omega_3 & 0\\ 0 & 0 & 0 & 2\xi_4\,\omega_4\end{bmatrix} \tag{14}$$

$$\hat{Q}^T K G_b = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} & \hat{q}_{31} & \hat{q}_{41} \\ \hat{q}_{12} & \hat{q}_{22} & \hat{q}_{32} & \hat{q}_{42} \\ \hat{q}_{13} & \hat{q}_{23} & \hat{q}_{33} & \hat{q}_{43} \\ \hat{q}_{14} & \hat{q}_{24} & \hat{q}_{34} & \hat{q}_{44} \end{bmatrix} \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix} \tag{15}$$

$$\hat{Q}^T K G_b = \begin{bmatrix} \hat{q}_{11} k_1 & \hat{q}_{21} k_2 & \hat{q}_{31} k_3 & \hat{q}_{41} k_4 \\ \hat{q}_{12} k_1 & \hat{q}_{22} k_2 & \hat{q}_{32} k_3 & \hat{q}_{42} k_4 \\ \hat{q}_{13} k_1 & \hat{q}_{23} k_2 & \hat{q}_{33} k_3 & \hat{q}_{43} k_4 \\ \hat{q}_{14} k_1 & \hat{q}_{24} k_2 & \hat{q}_{34} k_3 & \hat{q}_{44} k_4 \end{bmatrix} \tag{16}$$

A typical equation for modal coordinate i is

$$\ddot{\eta}_i + 2\xi_i \omega_i \dot{\eta}_i + \omega_i^2 \eta_i =$$

$$\begin{aligned} &\hat{q}_{1i}\, 2\xi_1 \omega_1 \dot{y}_1 + \hat{q}_{2i}\, 2\xi_2 \omega_2 \dot{y}_2 + \hat{q}_{3i}\, 2\xi_3 \omega_3 \dot{y}_3 + \hat{q}_{4i}\, 2\xi_4 \omega_4 k_4 \dot{y}_4 \\ &+ \hat{q}_{1i} k_1 y_1 + \hat{q}_{2i} k_2 y_2 + \hat{q}_{3i} k_3 y_3 + \hat{q}_{4i} k_4 y_4 \end{aligned} \tag{17}$$

Recall

$$I \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \\ \ddot{\eta}_3 \\ \ddot{\eta}_4 \end{bmatrix} + \hat{C} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \\ \dot{\eta}_4 \end{bmatrix} + \Omega \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} = \hat{Q}^T K G_b \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \tag{18}$$

The solution method for arbitrary inputs is given in Reference 3.

References

1. T. Irvine, Vibration Response of a Rigid Rectangular Plate or Frame Mounted on Four Springs, Vibrationdata, 2009.

2. T. Irvine, The Generalized Coordinate Method for Discrete Systems Subjected to Base Excitation, Vibrationdata, 2004.

3. T. Irvine, Modal Transient Response to Arbitrary Base Excitation via Convolution Revision A, Vibrationdata, 2009.

APPENDIX A

Let

$$\alpha = \frac{L_y}{L_1} \quad , \qquad \alpha' = \frac{1}{L_1} \quad , \qquad \lambda = 1 - \alpha \tag{A-1}$$

$$\beta = \frac{L_z}{L_2} \quad , \qquad \beta' = \frac{1}{L_2} \quad , \qquad \sigma = 1 - \beta \tag{A-2}$$

The global mass matrix MG is

$$\begin{aligned} MG = \quad & M \begin{bmatrix} \sigma^2\lambda^2 & \sigma\beta\lambda^2 & \alpha\beta\sigma\lambda & \alpha\lambda\sigma^2 \\ \sigma\beta\lambda^2 & \beta^2\lambda^2 & \alpha\beta^2\lambda & \alpha\beta\sigma\lambda \\ \alpha\beta\sigma\lambda & \alpha\beta^2\lambda & \alpha^2\beta^2 & \alpha^2\beta\sigma \\ \alpha\lambda\sigma^2 & \alpha\beta\sigma\lambda & \alpha^2\beta\sigma & \alpha^2\sigma^2 \end{bmatrix} \\ & + Jy(\beta')^2 \begin{bmatrix} \lambda^2 & -\lambda^2 & -\alpha\lambda & \alpha\lambda \\ -\lambda^2 & \lambda^2 & \alpha\lambda & -\alpha\lambda \\ -\alpha\lambda & \alpha\lambda & \alpha^2 & -\alpha^2 \\ \alpha\lambda & -\alpha\lambda & -\alpha^2 & \alpha^2 \end{bmatrix} \\ & + Jz(\alpha')^2 \begin{bmatrix} \sigma^2 & \sigma\beta & -\sigma\beta & -\sigma^2 \\ \sigma\beta & \beta^2 & -\beta^2 & -\sigma\beta \\ -\sigma\beta & -\beta^2 & \beta^2 & \sigma\beta \\ -\sigma^2 & -\sigma\beta & \sigma\beta & \sigma^2 \end{bmatrix} \end{aligned} \tag{A-3}$$

The global stiffness matrix KG is

$$KG = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix} \tag{A-4}$$

APPENDIX B

Example

Normal Modes Analysis

Consider a square plate (L x L) with the CG in the middle. All springs have the same stiffness k.

Assume a constant mass which is independent of the geometry. The length L is arbitrary because it will cancel out in the two inertia matrices.

Let

$$M = 1 \text{ lbm} = 0.00259 \text{ lbf sec^2/in} \tag{B-1}$$

$$K = 250 \text{ lbf/in} \tag{B-2}$$

Note that Jx = Jy = J for this example.

$$J \approx \frac{1}{12} M \left(L^2 \right) \tag{B-3}$$

$$J = 0.0833 \text{ lbm in^2} = 0.000216 \text{ lbf sec^2 in} \tag{B-4}$$

The global mass matrix is

$$MG = \frac{1}{16} M \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \frac{1}{4L^2} J \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} + \frac{1}{4L^2} J \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \tag{B-5}$$

$$MG = \frac{1}{16}M\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}+\frac{1}{4L^2}J\begin{bmatrix}2&0&-2&0\\0&2&0&-2\\-2&0&2&0\\0&-2&0&2\end{bmatrix}$$

(B-6)

$$MG = \frac{1}{16}M\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}+\frac{1}{48}M\begin{bmatrix}2&0&-2&0\\0&2&0&-2\\-2&0&2&0\\0&-2&0&2\end{bmatrix}$$

(B-7)

The global stiffness matrix is

$$KG = K\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

(B-8)

The calculation is completed via a Matlab code. The final, global mass and stiffness matrices are

```
MG =

  1.0e-003 *

    0.2700    0.1619    0.0538    0.1619
    0.1619    0.2700    0.1619    0.0538
    0.0538    0.1619    0.2700    0.1619
    0.1619    0.0538    0.1619    0.2700
```

(B-9)

```
KG =

   250     0     0     0
     0   250     0     0
     0     0   250     0
     0     0     0   250
```

(B-10)

Note that the mass unit is lbf sec^2/in. The stiffness unit is lbf/in.

The eigenvalues are

```
lambda =

  1.0e+006 *

    0.3860    1.1562    1.1562       Inf
```

(B-11)

```
  Natural Frequencies =

     98.88 Hz
     171.1 Hz
     171.1 Hz
       Inf Hz
```

(B-12)

The first three mode shapes in column format are

```
  -19.6469  -48.0864   -0.2209
  -19.6469   -0.0000   48.0859
  -19.6469   48.0864    0.2209
  -19.6469    0.0000  -48.0859
```

(B-13)

The fourth mode is

```
  1.0e+009 *

    4.2950
   -4.2950
    4.2950
   -4.2950
```

(B-14)

The fourth mode shape is fictitious. It implies that the rigid plate undergoes elastic deformation. The fourth natural frequency and its corresponding mode shape should be disregarded.

Hand Calculation Check

Translation

$$fn = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{B-15}$$

The total stiffness is 4 x 250 lbf/in

$$fn = \frac{1}{2\pi}\sqrt{\frac{1000 \text{ lbf / in}}{0.00259 \text{ lbf sec^2/in}}} \tag{B-16}$$

$$fn = 98.9 \text{ Hz} \tag{B-17}$$

The frequency agrees with the first natural frequency of the Matlab output.

Rotation

Assume that the length is 1 inch for simplicity. This natural frequency is effectively independent of this value, however, since the length cancels out from the numerator and denominator.

$$fn = \frac{1}{2\pi}\sqrt{\frac{k_r}{J}} \tag{B-18}$$

$$k_r = 4 \text{ x } 250 \text{ lbf/in x } (0.5 \text{ inch})\text{^}2 = 250 \text{ lbf in} \tag{B-19}$$

$$J = \frac{1}{12} M\left(L^2\right) \tag{B-20}$$

$$J = \frac{1}{12}(\ 1\ \text{lbm}\ )(\ 1\ \text{inch}\ )^2 = 0.0833\ \ \text{lbm in}^2 \tag{B-21}$$

$$J = 0.000216\ \text{lbf sec\^{}2 in} \tag{B-22}$$

$$fn = \frac{1}{2\pi}\sqrt{\frac{250\ \ \text{lbf in}}{0.000216\ \text{lbf sec\^{}2 in}}} = 171.3\ \ \text{Hz} \tag{B-23}$$

This value agrees with second and third natural frequencies from the Matlab output.

Modal Transient Analysis

Assume that Q = 10 for all modes.

Now apply base inputs. The following input example could represent a vehicle traveling over a bump in the road. Note that the fundamental frequency is much higher than that of an automobile, however.

The CG response is found via a Matlab script, plate_4_base.m. This script applies the method from Reference 3.

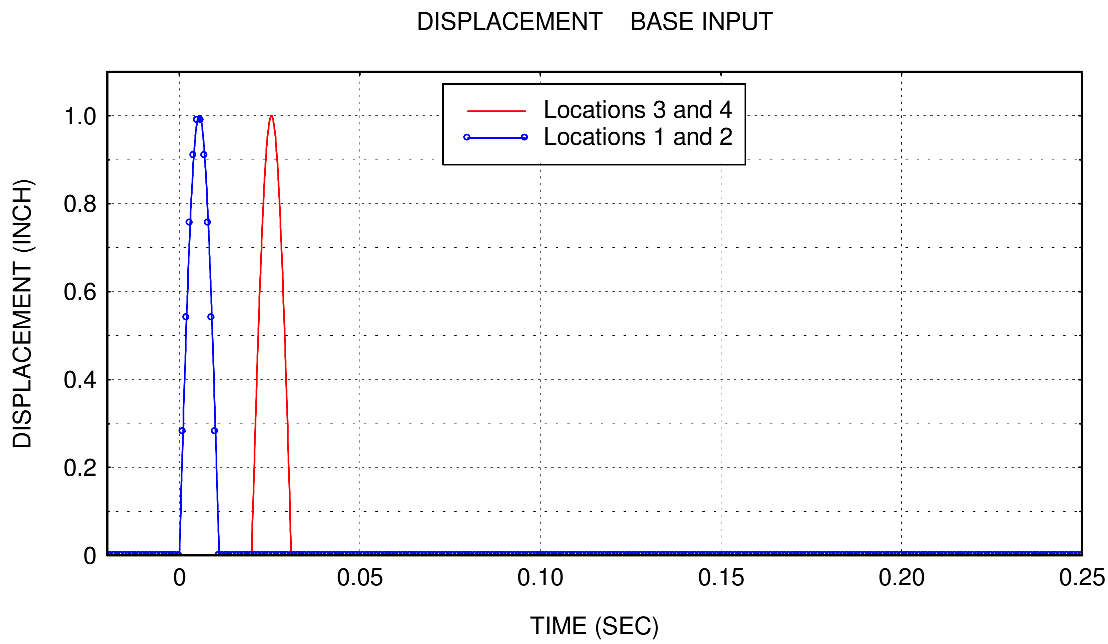


Figure B-1.

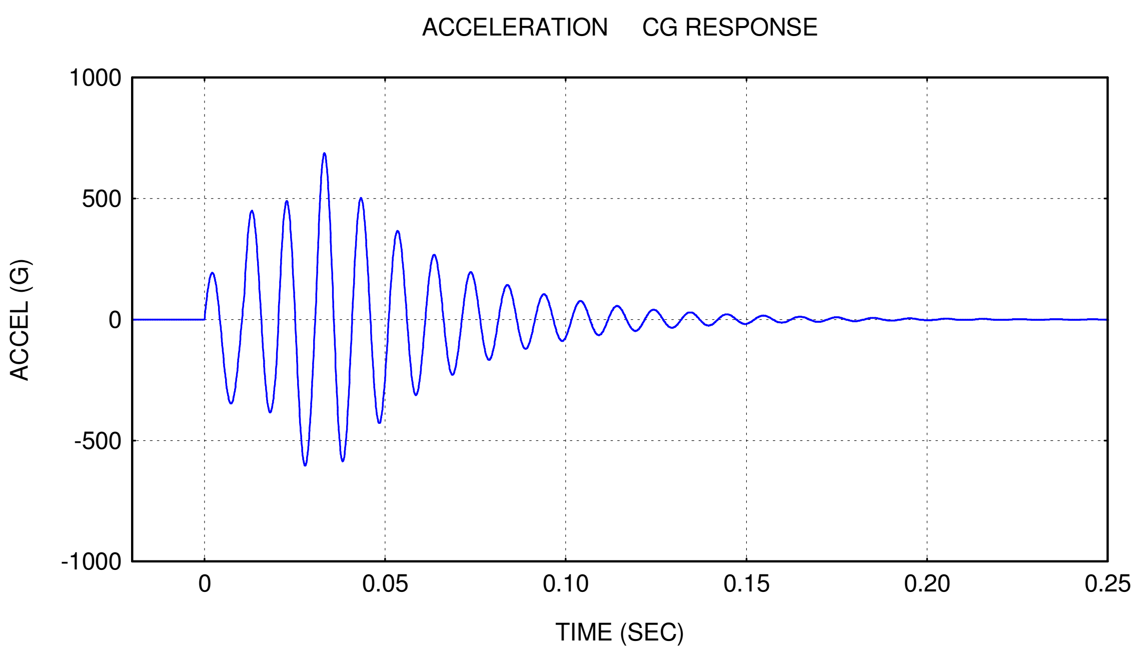
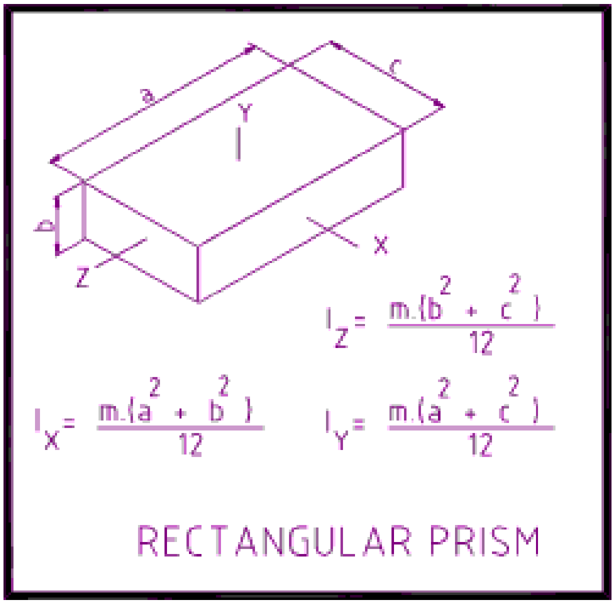


Figure B-2.

APPENDIX C

Mass Moment of Inertia



http://www.roymech.co.uk/Useful_Tables/Form/Dynamics_inertia.html

$$J \approx \frac{1}{12} M\left(L^2\right) \quad \text{for a thin plate} \tag{C-1}$$