FREE VIBRATION OF A CANTILEVER BEAM

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Equation of Motion

Consider the cantilever beam in Figure 1 undergoing transverse, free vibration.

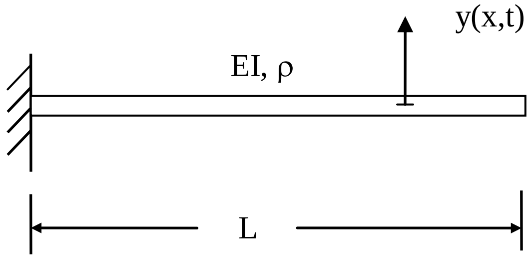

Figure 1.

The following equations are taken from Reference 1.

The governing differential equation is

$$EI\frac{\partial^4 y}{\partial x^4}+\rho\frac{\partial^2 y}{\partial t^2}=-0 \tag{1}$$

where

E	is the modulus of elasticity
I	is the area moment of inertia
L	is the length
ρ	is the mass density (mass/length)

The natural frequencies are

$$\omega_n = \beta_n^{\,2} \sqrt{EI / \rho} \tag{2}$$

The eigenvalues are

n	$\beta_n L$
1	1.875104
2	4.69409
3	7.85476
4	10.99554
5	(2n-1)π/2

The first four eigenvectors normalized to an amplitude of 1 are

$$Y_1(x) = C_1 \left[\cosh(\beta_1 x) - \cos(\beta_1 x)\right] - 0.73410 \left[\sinh(\beta_1 x) - \sin(\beta_1 x)\right] \tag{3}$$

$$Y_2(x) = C_2 \left[\cosh(\beta_2 x) - \cos(\beta_2 x)\right] - 1.01847 \left[\sinh(\beta_2 x) - \sin(\beta_2 x)\right] \tag{4}$$

$$Y_3(x) = C_3 \left[\cosh(\beta_3 x) - \cos(\beta_3 x)\right] - 0.99922 \left[\sinh(\beta_3 x) - \sin(\beta_3 x)\right] \tag{5}$$

$$Y_4(x) = C_4 \left[\cosh(\beta_4 x) - \cos(\beta_4 x)\right] - 1.00003 \left[\sinh(\beta_4 x) - \sin(\beta_4 x)\right] \tag{6}$$

where

$$C_1 = 1/\left\{\left[\cosh(\beta_1 L) - \cos(\beta_1 L)\right] - 0.73410 \left[\sinh(\beta_1 L) - \sin(\beta_1 L)\right]\right\} \tag{7}$$

$$C_2 = \left[\cosh(\beta_2 L) - \cos(\beta_2 L)\right] - 1.01847 \left[\sinh(\beta_2 L) - \sin(\beta_2 L)\right] \tag{8}$$

$$C_3 = [\cosh(\beta_3 L) - \cos(\beta_3 L)] - 0.99922\ [\sinh(\beta_3 L) - \sin(\beta_3 L)] \tag{9}$$

$$C_4 = [\cosh(\beta_4 L) - \cos(\beta_4 L)] - 1.00003\ [\sinh(\beta_4 L) - \sin(\beta_4 L)] \tag{10}$$

The total displacement is

$$y(x,t) = \sum_{n=1}^{\infty} \left\{ Y_n(x)\ [a_n \sin(\omega_n t) + b_n \cos(\omega_n t)]\ \right\} \tag{11}$$

$$\dot{y}(x,t) = \sum_{n=1}^{\infty} \left\{ \omega_n Y_n(x)\ [a_n \cos(\omega_n t) - b_n \sin(\omega_n t)]\ \right\} \tag{12}$$

For zero initial velocity $a_n = 0$

$$y(x,t) = \sum_{n=1}^{\infty} \left\{ b_n\ Y_n(x) \cos(\omega_n t)\ \right\} \tag{13}$$

Let the initial displacement be equal to the fundamental mode shape scaled to a displacement of 1 at the free end.

$$y(x,0) = Y_1(x) \tag{14}$$

Premultiply by $Y_m(x)$ and integrate.

$$\int_0^L Y_1(x)\ Y_m(x)\ dx = \sum_{n=1}^{\infty} \left\{ b_n \int_0^L Y_n(x) Y_m(x)\, dx \right\} \tag{15}$$

For m ≠ n,

The integral on the right hand side of (15) goes to zero. The steps are omitted for brevity.

For m = n,

$$\int_0^L Y_1(x)\, Y_n(x)\, dx = \sum_{n=1}^{\infty}\left\{ b_n \int_0^L Y_n^{\,2}(x)\, dx \right\} \tag{16}$$

For n = 1,

$$\int_0^L Y_1^{\,2}(x)\; dx = b_1 \int_0^L Y_1^{\,2}(x)\, dx \tag{17}$$

$$b_1 = 1 \tag{18}$$

$$y(x,t) = Y_1(x)\, \cos(\omega_1 t) \tag{19}$$

$$y(x,t) = C_1 \{ [\cosh(\beta_1 x) - \cos(\beta_1 x)] - 0.73410\; [\sinh(\beta_1 x) - \sin(\beta_1 x)] \} \cos(\omega_1 t) \tag{20}$$

$$\frac{\partial}{\partial x} y(x,t) = \beta_1 C_1 \{ [\sinh(\beta_1 x) + \sin(\beta_1 x)] - 0.73410\; [\cosh(\beta_1 x) - \cos(\beta_1 x)] \} \cos(\omega_1 t) \tag{21}$$

$$\frac{\partial^2}{\partial x^2} y(x,t) = \beta_1^{\,2} C_1 \{ [\cosh(\beta_1 x) + \cos(\beta_1 x)] - 0.73410\; [\sinh(\beta_1 x) + \sin(\beta_1 x)] \} \cos(\omega_1 t) \tag{22}$$

$$\frac{\partial^3}{\partial x^3} y(x,t) = \beta_1^{\,3} C_1 \{ [\sinh(\beta_1 x) - \sin(\beta_1 x)] - 0.73410\; [\cosh(\beta_1 x) + \cos(\beta_1 x)] \} \cos(\omega_1 t) \tag{23}$$

Example

Consider a beam with the following properties:

Cross-Section	Circular
Boundary Conditions	Fixed-Free
Material	Aluminum

Length	L	=	24 inch
Diameter	D	=	1 inch
Area	A	=	0.785 inch^2
Area Moment of Inertia	I	=	0.0491 inch^4
Elastic Modulus	E	=	1.0e+07 lbf/in^2
Mass Density	ρ	=	0.1 lbm/in^3
Speed of Sound in Material	c	=	1.96e+05 in/sec
Viscous Damping Ratio	ξ	=	0

The beam is subjected to an initial displacement equal to its first mode shape, scaled so that the displacement at the free end is 0.010 inch. The initial velocity is zero.

The normal modes results are:

Table 1. Natural Frequency Results, Beam Fixed-Free		
Mode	fn (Hz)	Effective Modal Mass (lbm)
1	47.7	1.15
2	299	0.36
3	837	0.12
4	1641	0.06

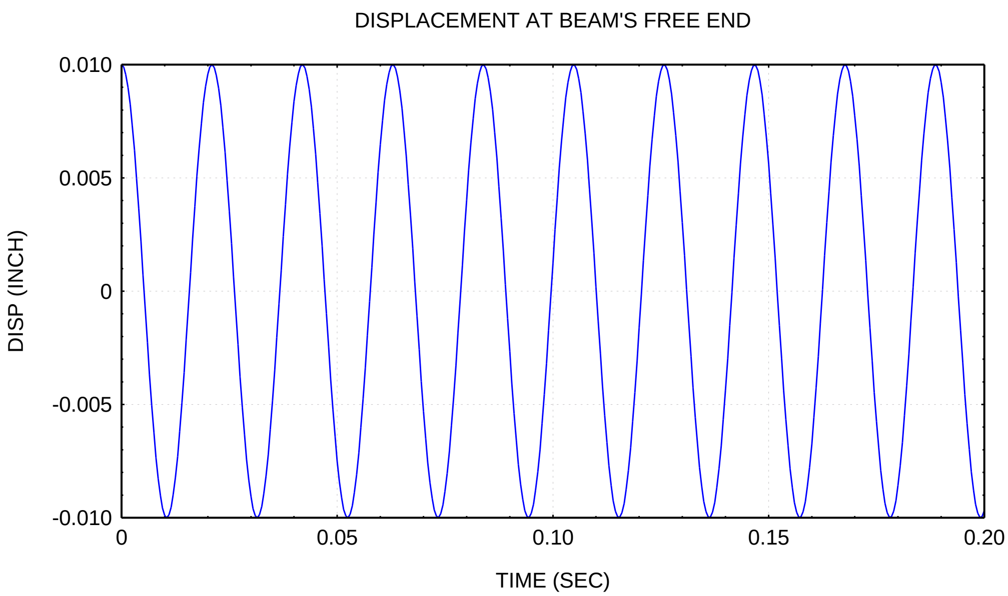

Figure 2.

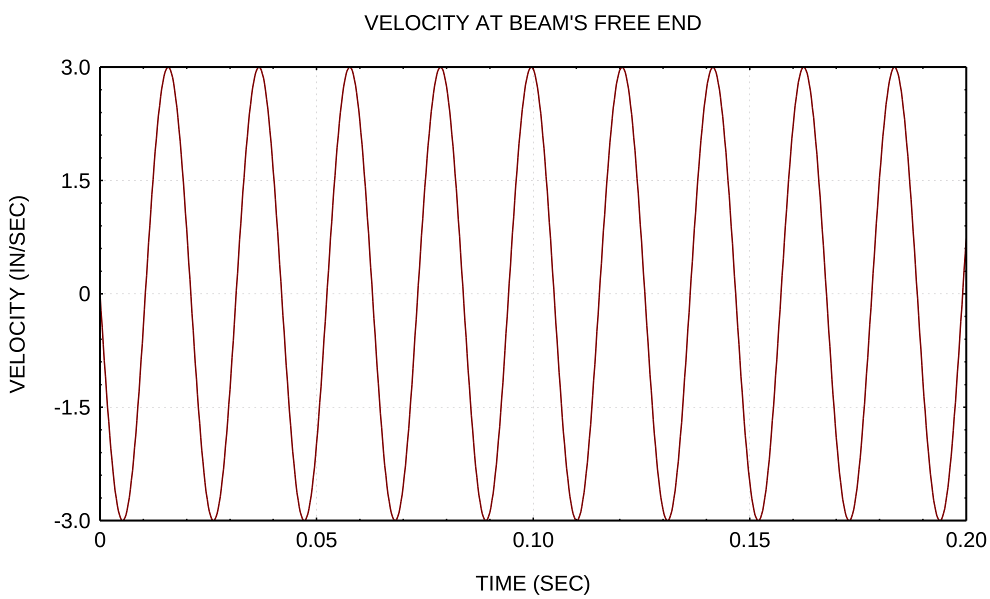

Figure 3.

The bending moment M(x,t) is

$$M(x,t) = EI\ C_1 \frac{\partial^2}{\partial x^2} y(x,t) \tag{24}$$

$$M(x,t) = EI\ \beta_1{}^2\, C_1\, D\left\{\left[\cosh(\beta_1 x) + \cos(\beta_1 x)\right] - 0.73410\ \left[\sinh(\beta_1 x) + \sin(\beta_1 x)\right]\right\} \cos(\omega_1 t) \tag{25}$$

The maximum bending moment occurs at the fixed end.

$$M_{max} = 2\, EI\, \beta_1{}^2 C_1\, D \tag{26}$$

where D is the initial displacement at the free end.

$$M_{max} = 2\left(1.0e+07\ \text{lbf/in\^{}2}\right)(0.0491\ \text{in\^{}4})\left(\frac{1.875104}{24\,\text{in}}\right)^2\left(\frac{1}{2}\right)(0.010\ \text{in}) \tag{27}$$

$$M_{max} = 29.964\ \ \text{in lbf} \tag{28}$$

The shear force V(x,t) is

$$V(x,t) = EI \frac{\partial^3}{\partial x^3} y(x,t) \tag{29}$$

$$V(x,t) = EI\, \beta_1{}^3\, C_1 D\left\{\left[\sinh(\beta_1 x) - \sin(\beta_1 x)\right] - 0.73410\ \left[\cosh(\beta_1 x) + \cos(\beta_1 x)\right]\right\} \cos(\omega_1 t) \tag{30}$$

The maximum shear occurs at the fixed end.

$$V(x,t) = (1.0e+07 \ \text{lbf/in\^{}2})(0.0491 \ \text{in\^{}4})\left(\frac{1.875104}{24\,\text{in}}\right)^3 (0.010 \ \text{in})(2)\left(\frac{1}{2}\right)(-0.73410) \tag{31}$$

$$V_{max} = -1.719 \ \text{lbf} \tag{32}$$

<u>References</u>

1. T. Irvine, Bending Frequencies of Beams, Rod, and Pipes, Revision M, Vibrationdata, 2010.