THE TIME-DOMAIN RESPONSE OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM SUBJECTED TO A SINUSOIDAL FORCE WITH PHASE

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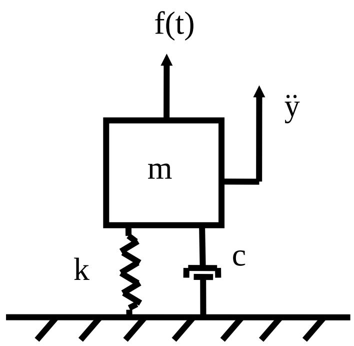
Consider a single-degree-of-freedom system.

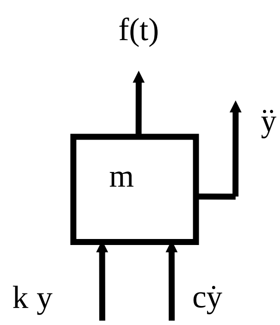


where

m	is the mass
c	is the viscous damping coefficient
k	k is the stiffness
y	is the absolute displacement of the mass
f(t)	is the applied force

Note that the double-dot denotes acceleration.

The free-body diagram is



Summation of forces in the vertical direction

$$\sum F = m\ddot{y} \tag{1}$$

$$m\ddot{y} = -c\dot{y} - ky + f(t) \tag{2}$$

$$m\ddot{y} + c\dot{y} + ky = f(t) \tag{3}$$

Divide through by m,

$$\ddot{y} + \left(\frac{c}{m}\right)\dot{y} + \left(\frac{k}{m}\right)y = \left(\frac{1}{m}\right)f(t) \tag{4}$$

By convention,

$$(c/m) = 2\xi\omega_n \tag{5}$$

$$(k/m) = \omega_n^{\,2} \tag{6}$$

where

ω_n is the natural frequency in (radians/sec)
ξ is the damping ratio

By substitution,

$$\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = \frac{1}{m}f(t) \tag{7}$$

Now assume a sinusoidal force function.

$$f(t) = f_o \sin(\omega t + \psi) \tag{8}$$

The governing equation becomes.

$$\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 x = \frac{1}{m}f_o \sin(\omega t + \psi) \tag{9}$$

The right-hand-side can be rewritten as

$$\ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 y = \frac{\omega_n^2}{k} f_o \sin(\omega t + \psi) \tag{10}$$

Take the Laplace transform of each side.

$$L\left\{\ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 y\right\} = L\left\{\frac{\omega_n^2}{k} f_o \sin(\omega t + \psi)\right\} \tag{11}$$

$$\begin{aligned} &s^2 Y(s) - s y(0) - y'(0) \\ &+ 2\xi\omega_n s Y(s) - 2\xi\omega_n y(0) \\ &+ \omega_n^2 Y(s) = \frac{\omega_n^2}{k} f_o \left\{\frac{\sin(\psi)s + \cos(\psi)\omega}{s^2 + \omega^2}\right\} \end{aligned} \tag{12}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\} Y(s) - \left\{s + 2\xi\omega_n\right\} y(0) - y'(0) = \frac{\omega_n^2}{k} f_o \left\{\frac{\sin(\psi)s + \cos(\psi)\omega}{s^2 + \omega^2}\right\} \tag{13}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\} Y(s) = \frac{\omega_n^2}{k} f_o \left\{\frac{\sin(\psi)s + \cos(\psi)\omega}{s^2 + \omega^2}\right\} + \left\{s + 2\xi\omega_n\right\} y(0) + y'(0) \tag{14}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 - (\xi\omega_n)^2 + \omega_n^2 \tag{15}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_n^2\left(1 - \xi^2\right) \tag{16}$$

Let

$$\omega_d = \omega_n \sqrt{1 - \xi^2} \tag{17}$$

Substitute equation (17) into (16).

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_d^2 \tag{18}$$

$$\left\{(s + \xi\omega_n)^2 + \omega_d^2\right\} Y(s) = \frac{\omega_n^2}{k} f_o \left\{\frac{\sin(\psi)s + \cos(\psi)\omega}{s^2 + \omega^2}\right\} + \{s + 2\xi\omega_n\} y(0) + y'(0) \tag{19}$$

$Y(s) =$

$$\frac{\omega_n^2}{k} f_o \left\{\frac{\sin(\psi)s + \cos(\psi)\omega}{s^2 + \omega^2}\right\}\left\{\frac{1}{(s + \xi\omega_n)^2 + \omega_d^2}\right\}$$
$$+\left\{\frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} y(0) + \left\{\frac{1}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} y'(0) \tag{20}$$

Divide the right-hand-side of equation (20) into two parts.

$$Y(s) = Y_1(s) + Y_2(s) \tag{21}$$

where

$$Y_1(s) = \frac{\omega_n^2}{k} f_o \left\{\frac{\sin(\psi)s + \cos(\psi)\omega}{s^2 + \omega^2}\right\}\left\{\frac{1}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} \tag{22}$$

$$Y_2(s) = \left\{\frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} y(0) + \left\{\frac{1}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} y'(0) \tag{23}$$

Consider $Y_1(s)$ from equation (22).

$$Y_1(s) = \frac{\omega\,\omega_n^{\,2}}{k} f_o \left\{\frac{1}{s^2+\omega^2}\right\}\left\{\frac{1}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right\} \tag{24}$$

Expand into partial fractions.

$$\left\{\frac{\sin(\psi)s+\cos(\psi)\omega}{s^2+\omega^2}\right\}\left\{\frac{1}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right\} = \left\{\frac{\lambda s+\rho}{s^2+\omega^2}\right\}+\left\{\frac{\sigma s+\phi}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right\} \tag{25}$$

Take the inverse Laplace transform of the first term on the right-hand-side of equation (25).

$$L^{-1}\left\{\frac{\lambda s+\rho}{s^2+\omega^2}\right\} = L^{-1}\left\{\lambda\left[\frac{s}{s^2+\omega^2}\right]+\rho\left[\frac{1}{s^2+\omega^2}\right]\right\} \tag{26}$$

$$L^{-1}\left\{\frac{\lambda s+\rho}{s^2+\omega^2}\right\} = L^{-1}\left\{\lambda\left[\frac{s}{s^2+\omega^2}\right]+\left[\frac{\rho}{\omega}\right]\left[\frac{\omega}{s^2+\omega^2}\right]\right\} \tag{27}$$

The inverse transform is obtained from standard tables.

$$L^{-1}\left\{\frac{\lambda s+\rho}{s^2+\omega^2}\right\} = \lambda\cos(\omega t)+\left[\frac{\rho}{\omega}\right]\sin(\omega t) \tag{28}$$

Now take the inverse Laplace transform of the second term on the right-hand-side of equation (25).

$$L^{-1}\left\{\frac{\sigma s+\phi}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right\}=L^{-1}\left\{\frac{\sigma s}{(s+\xi\omega_n)^2+\omega_d^{\,2}}+\frac{\phi}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right\} \tag{29}$$

$$L^{-1}\left\{\frac{\sigma s+\phi}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right\}=$$

$$L^{-1}\left\{[\sigma]\left[\frac{s}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right]+\left[\frac{\phi}{\omega_d}\right]\left[\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right]\right\} \tag{30}$$

$$L^{-1}\left\{\frac{\sigma s+\phi}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right\}=$$

$$L^{-1}\left\{[\sigma]\left[\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right]-[\sigma]\left[\frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right]+\left[\frac{\phi}{\omega_d}\right]\left[\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^{\,2}}\right]\right\} \tag{31}$$

$$L^{-1}\left\{\frac{\sigma s+\phi}{(s+\xi\omega_n)^2+\omega_d^2}\right\}=$$

$$L^{-1}\left\{[\sigma]\left[\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right]-\left[\frac{\xi\omega_n\sigma}{\omega_d}\right]\left[\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}\right]+\left[\frac{\phi}{\omega_d}\right]\left[\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}\right]\right\}$$

(32)

$$L^{-1}\left\{\frac{\sigma s+\phi}{(s+\xi\omega_n)^2+\omega_d^2}\right\}=$$

$$L^{-1}\left\{[\sigma]\left[\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right]+\left[\frac{\phi-\xi\omega_n\sigma}{\omega_d}\right]\left[\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}\right]\right\}$$

(33)

The inverse transform is obtained from standard tables.

$$L^{-1}\left\{\frac{\sigma s+\phi}{(s+\xi\omega_n)^2+\omega_d^2}\right\}= e^{-\xi\omega_n t}\left\{\sigma\cos(\omega_d t)+\left[\frac{\phi-\xi\omega_n\sigma}{\omega_d}\right]\sin(\omega_d t)\right\}$$

(34)

In summary,

$$L^{-1}\left\{\left[\frac{}{s^2+\omega^2}\right]\left[\frac{1}{(s+\xi\omega_n)^2+\omega_d^2}\right]\right\}$$

$$=\lambda\cos(\omega t)+\left[\frac{\rho}{\omega}\right]\sin(\omega t)+e^{-\xi\omega_n t}\left\{\sigma\cos(\omega_d t)+\left[\frac{\phi-\xi\omega_n\sigma}{\omega_d}\right]\sin(\omega_d t)\right\}$$

(35)

Assemble equations (28) and (35).

$$y_1(t)\quad=\left\{\frac{\omega\,\omega_n^2 f_o}{k}\right\}\left\{\lambda\cos(\omega t)+\left[\frac{\rho}{\omega}\right]\sin(\omega t)+e^{-\xi\omega_n t}\left\{\sigma\cos(\omega_d t)+\left[\frac{\phi-\xi\omega_n\sigma}{\omega_d}\right]\sin(\omega_d t)\right\}\right\}$$

(36)

Solve for the coefficients. Recall the partial fraction expansion.

$$\left\{\frac{\sin(\psi)s+\cos(\psi)\omega}{s^2+\omega^2}\right\}\left\{\frac{1}{(s+\xi\omega_n)^2+\omega_d^2}\right\}\quad=\left\{\frac{\lambda s+\rho}{s^2+\omega^2}\right\}+\left\{\frac{\sigma s+\phi}{(s+\xi\omega_n)^2+\omega_d^2}\right\}$$

(37)

Change the denominator to an equivalent form via equation (18).

$$\left\{\frac{\sin(\psi)s+\cos(\psi)\omega}{s^2+\omega^2}\right\}\left\{\frac{1}{s^2+2\xi\omega_n s+\omega_n^2}\right\}\quad=\left\{\frac{\lambda s+\rho}{s^2+\omega^2}\right\}+\left\{\frac{\sigma s+\phi}{s^2+2\xi\omega_n s+\omega_n^2}\right\}$$

(38)

$$\sin(\psi)s+\cos(\psi)\omega \quad = \{\lambda s+\rho\}\left\{s^2+2\xi\omega_n s+\omega_n^2\right\}+\{\sigma s+\phi\}\left\{s^2+\omega^2\right\} \tag{39}$$

$$\begin{aligned}\sin(\psi)s+\cos(\psi)\omega \quad &= \lambda s^3+(\rho+2\xi\omega_n\lambda)s^2+\left(2\xi\omega_n\rho+\lambda\omega_n^2\right)s+\left(\rho\omega_n^2\right)\\ &+\sigma s^3+\phi s^2+\sigma\omega^2 s+\phi\omega^2\end{aligned} \tag{40}$$

$$\begin{aligned}\sin(\psi)s+\cos(\psi)\omega \quad =&\\ &[\lambda+\sigma]s^3\\ &+[\rho+2\xi\omega_n\lambda+\phi]s^2\\ &+\left[2\xi\omega_n\rho+\lambda\omega_n^2+\sigma\omega^2\right]s\\ &+\left[\rho\omega_n^2+\phi\omega^2\right]\end{aligned} \tag{41a}$$

Equation (41a) implies four separate equations.

$$\lambda+\sigma=0 \tag{41b}$$

$$\rho+2\xi\omega_n\lambda+\phi=0 \tag{41c}$$

$$2\xi\omega_n\rho+\lambda\omega_n^2+\sigma\omega^2=\sin(\psi) \tag{41d}$$

$$\rho\omega_n^2+\phi\omega^2=\cos(\psi)\omega \tag{41e}$$

Equations (41b) through (41e) can be assembled into matrix form.

$$\begin{bmatrix}1 & 0 & 1 & 0\\ 2\xi\omega_n & 1 & 0 & 1\\ \omega_n^2 & 2\xi\omega_n & \omega^2 & 0\\ 0 & \omega_n^2 & 0 & \omega^2\end{bmatrix}\begin{bmatrix}\lambda\\ \rho\\ \sigma\\ \phi\end{bmatrix}=\begin{bmatrix}0\\ 0\\ \sin(\psi)\\ \omega\cos(\psi)\end{bmatrix} \tag{42}$$

Gaussian elimination is used to simplify the coefficient matrix.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2\xi\omega_n & 1 \\ 0 & 2\xi\omega_n & \omega^2-\omega_n^2 & 0 \\ 0 & \omega_n^2 & 0 & \omega^2 \end{bmatrix} \begin{bmatrix} \lambda \\ \rho \\ \sigma \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sin(\psi) \\ \omega\cos(\psi) \end{bmatrix} \tag{43}$$

Equation (43) can be reduced to a 3 x 3 matrix.

$$\begin{bmatrix} 1 & -2\xi\omega_n & 1 \\ 2\xi\omega_n & \omega^2-\omega_n^2 & 0 \\ \omega_n^2 & 0 & \omega^2 \end{bmatrix} \begin{bmatrix} \rho \\ \sigma \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ \sin(\psi) \\ \omega\cos(\psi) \end{bmatrix} \tag{44}$$

Complete the solution using Cramer’s rule.

$$\det\begin{bmatrix} 1 & -2\xi\omega_n & 1 \\ 2\xi\omega_n & \omega^2-\omega_n^2 & 0 \\ \omega_n^2 & 0 & \omega^2 \end{bmatrix} = \omega^2\left(\omega^2-\omega_n^2\right) + \omega^2(2\xi\omega_n)^2 - \omega_n^2\left(\omega^2-\omega_n^2\right) \tag{45}$$

$$\det\begin{bmatrix} 1 & -2\xi\omega_n & 1 \\ 2\xi\omega_n & \omega^2-\omega_n^2 & 0 \\ \omega_n^2 & 0 & \omega^2 \end{bmatrix} = \left(\omega^2-\omega_n^2\right)^2 + (2\xi\omega\omega_n)^2 \tag{46}$$

$$\rho = \frac{1}{\left(\omega^2-\omega_n^2\right)^2 + (2\xi\omega\omega_n)^2}\det\begin{bmatrix} 0 & -2\xi\omega_n & 1 \\ \sin(\psi) & \omega^2-\omega_n^2 & 0 \\ \omega\cos(\psi) & 0 & \omega^2 \end{bmatrix} \tag{47}$$

$$\rho = \frac{2\xi\omega_n \omega^2 \sin(\psi) - \left(\omega^2 - \omega_n^2\right)\omega \cos(\psi)}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\,\omega_n)^2} \tag{48}$$

$$\sigma = \frac{1}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\,\omega_n)^2} \det\begin{bmatrix} 1 & 0 & 1 \\ 2\xi\omega_n & \sin(\psi) & 0 \\ \omega_n^2 & \omega\cos(\psi) & \omega^2 \end{bmatrix} \tag{49}$$

$$\sigma = \frac{\left(\omega^2 - \omega_n^2\right)\sin(\psi) + 2\xi\omega_n \omega \cos(\psi)}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\,\omega_n)^2} \tag{50}$$

Recall equation (41b).

$$\lambda = -\sigma \tag{51}$$

$$\lambda = \frac{-\left[\left(\omega^2 - \omega_n^2\right)\sin(\psi) + 2\xi\omega_n \omega \cos(\psi)\right]}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\,\omega_n)^2} \tag{52}$$

$$\phi = \frac{1}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\,\omega_n)^2} \det\begin{bmatrix} 1 & -2\xi\omega_n & 0 \\ 2\xi\omega_n & \omega^2 - \omega_n^2 & \sin(\psi) \\ \omega_n^2 & 0 & \omega\cos(\psi) \end{bmatrix} \tag{53}$$

$$\phi = \frac{\left(\omega^2 - \omega_n^2 + (2\xi\omega_n)^2\right)\omega\cos(\psi) - 2\xi\omega_n{}^3\sin(\psi)}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2} \tag{54}$$

The coefficients are summarized in equation (55).

$$\begin{bmatrix}\lambda\\ \rho\\ \sigma\\ \phi\end{bmatrix} = \frac{1}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2}\begin{bmatrix} -\left[\left(\omega^2 - \omega_n^2\right)\sin(\psi) + 2\xi\omega_n\omega\cos(\psi)\right] \\ 2\xi\omega_n\omega^2\sin(\psi) - \left(\omega^2 - \omega_n^2\right)\omega\cos(\psi) \\ \left[\left(\omega^2 - \omega_n^2\right)\sin(\psi) + 2\xi\omega_n\omega\cos(\psi)\right] \\ \left(\omega^2 - \omega_n^2 + (2\xi\omega_n)^2\right)\omega\cos(\psi) - 2\xi\omega_n{}^3\sin(\psi) \end{bmatrix} \tag{55}$$

Recall equation (23).

$$Y_2(s) = \left\{\frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}\right\}y(0) + \left\{\frac{1}{(s + \xi\omega_n)^2 + \omega_d^2}\right\}y'(0) \tag{56}$$

$$Y_2(s) = \left\{\frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}\right\}y(0) + \left\{\frac{\left[\frac{\xi\omega_n}{\omega_d}y(0) + \frac{1}{\omega_d}y'(0)\right]\omega_d}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} \tag{57}$$

The inverse Laplace transform from standard tables is

$$y_2(t) = [y(0)]\exp(-\xi\omega_n t)\cos(\omega_d t) + \left[\frac{\xi\omega_n y(0) + y'(0)}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t) \tag{58}$$

$$\begin{aligned} y_2(t) &= y(0)\exp(-\xi\omega_n t)\left\{\cos(\omega_d t) + \left[\frac{\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\} \\ &\quad + y'(0)\left[\frac{1}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t) \end{aligned} \tag{59}$$

The final displacement solution is obtained by adding equations (36) and (59).

$$y(t) = y_1(t) + y_2(t) \tag{60}$$

$$\begin{aligned} y(t) &= y(0)\, e^{-\xi\omega_n t}\left\{\cos(\omega_d t) + \left[\frac{\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\} \\ &\quad + y'(0)\left[\frac{1}{\omega_d}\right] e^{-\xi\omega_n t}\sin(\omega_d t) \\ &\quad + \left\{\frac{{\omega_n}^2 f_o}{k}\right\}\left\{\lambda\cos(\omega t) + \left[\frac{\rho}{\omega}\right]\sin(\omega t)\right\} \\ &\quad + \left\{\frac{{\omega_n}^2 f_o}{k}\right\}\left\{e^{-\xi\omega_n t}\right\}\left\{\sigma\cos(\omega_d t) + \left[\frac{\phi - \xi\omega_n\sigma}{\omega_d}\right]\sin(\omega_d t)\right\} \end{aligned} \tag{61}$$

The steady-state response $y_{ss}(t)$ is

$$y_{ss}(t) = \left\{\frac{\omega_n{}^2 f_o}{k}\right\}\left\{\lambda\cos(\omega t) + \left[\frac{\rho}{\omega}\right]\sin(\omega t)\right\} \tag{62}$$

$$y_{ss}(t) = \frac{\omega_n{}^2 f_o / k}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\,\omega_n)^2}$$
$$\left\{-\left[\left(\omega^2 - \omega_n^2\right)\sin(\psi) + 2\xi\omega_n\,\omega\cos(\psi)\right]\cos(\omega t) + \left[\frac{2\xi\omega_n\,\omega^2\sin(\psi) - \left(\omega^2 - \omega_n^2\right)\omega\cos(\psi)}{\omega}\right]\sin(\omega t)\right\}$$

(63)

$$y_{ss}(t) = \frac{\omega_n{}^2 f_o / k}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\,\omega_n)^2}\Bigg\{$$
$$-\left[\left(\omega^2 - \omega_n^2\right)\sin(\psi) + 2\xi\omega_n\,\omega\cos(\psi)\right]\cos(\omega t) + \left[2\xi\omega_n\,\omega\sin(\psi) - \left(\omega^2 - \omega_n^2\right)\cos(\psi)\right]\sin(\omega t)\Bigg\}$$

(64)

The magnitude is

$$\| y_{ss}(t) \| = \frac{\omega_n{}^2 f_o / k}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\,\omega_n)^2} \left\{ \sqrt{\left\{-\left[\left(\omega^2 - \omega_n^2\right)\sin(\psi) + 2\xi\omega_n\omega\cos(\psi)\right]\right\}^2 + \left\{+\left[2\xi\omega_n\omega\sin(\psi) - \left(\omega^2 - \omega_n^2\right)\cos(\psi)\right]\right\}^2} \right\} \tag{65}$$

The magnitude is

$$\| y_{ss}(t) \| = \frac{\omega_n{}^2 f_o / k}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\,\omega_n)^2} \sqrt{P} \tag{66}$$

$$P = \left\{\left(\omega^2 - \omega_n^2\right)\sin(\psi) + 2\xi\omega_n\omega\cos(\psi)\right\}^2 + \left\{2\xi\omega_n\omega\sin(\psi) - \left(\omega^2 - \omega_n^2\right)\cos(\psi)\right\}^2 \tag{67}$$

$$\begin{aligned} P = {} & \left(\omega^2 - \omega_n^2\right)^2 \sin^2(\psi) + 4\xi\omega_n\omega\left(\omega^2 - \omega_n^2\right)\cos(\psi)\sin(\psi) + 4\xi^2\omega_n{}^2\omega^2\cos^2(\psi) \\ & + \left(\omega^2 - \omega_n^2\right)^2 \cos^2(\psi) - 4\xi\omega_n\omega\left(\omega^2 - \omega_n^2\right)\cos(\psi)\sin(\psi) + 4\xi^2\omega_n{}^2\omega^2\sin^2(\psi) \end{aligned} \tag{68}$$

$$P = \left(\omega^2 - \omega_n^2\right)^2 + 4\xi^2\omega_n{}^2\omega^2 \tag{69}$$

$$P = \left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega_n\omega)^2 \tag{70}$$

$$\| y_{ss}(t) \| = \frac{\omega_n{}^2 f_o / k}{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\,\omega_n)^2} \sqrt{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega_n\omega)^2} \tag{71}$$

$$\| y_{ss}(t) \| = \frac{\omega_n{}^2 f_o / k}{\sqrt{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega_n\omega)^2}} \tag{72}$$

$$\| y_{ss}(t) \| = \frac{f_o / m}{\sqrt{\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega_n\omega)^2}} \tag{73}$$