THE IMPULSE RESPONSE FUNCTION

By Tom Irvine
Email: tomirvine@aol.com

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Introduction

Consider a single-degree-of-freedom system.

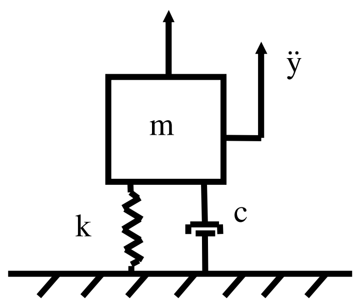

Figure 1.

The variables are

m	is the mass
c	is the viscous damping coefficient
k	is the stiffness
y	is the absolute displacement of the mass
f(t)	is the applied force

The free-body diagram is

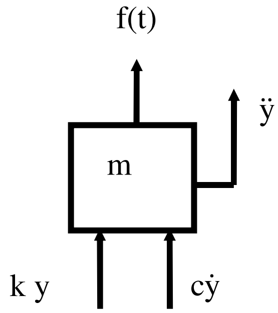

Figure 2.

Note that the double-dot denotes acceleration.

Summation of forces in the vertical direction

$$\sum F = m\ddot{y} \tag{1}$$

$$m\ddot{y} = -c\dot{y} - ky + f(t) \tag{2}$$

$$m\ddot{y} + c\dot{y} + ky = f(t) \tag{3}$$

Divide through by m,

$$\ddot{y} + \left(\frac{c}{m}\right)\dot{y} + \left(\frac{k}{m}\right)y = \left(\frac{1}{m}\right)f(t) \tag{4}$$

By convention,

$$(c/m) = 2\xi\omega_n \tag{5}$$

$$(k/m) = \omega_n{}^2 \tag{6}$$

where

ω_n is the natural frequency in (radians/sec)
ξ is the damping ratio

By substitution,

$$\ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 y = \frac{1}{m} f(t) \tag{7}$$

Now assume an impulse force using the Dirac delta function.

$$f(t) = \delta(t) \tag{8}$$

The governing equation becomes.

$$\ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 x = \frac{1}{m} \delta(t) \tag{9}$$

The right-hand-side can be rewritten as

$$\ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 y = \frac{\omega_n^2}{k} \delta(t) \tag{10}$$

Take the Laplace transform of each side.

$$L\left\{\ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 y\right\} = L\left\{\frac{\omega_n^2}{k} \delta(t)\right\} \tag{11}$$

$$\begin{aligned} &s^2 Y(s) - sy(0) - y'(0) \\ &+ 2\xi\omega_n sY(s) - 2\xi\omega_n y(0) \\ &+ \omega_n^2 Y(s) = \frac{\omega_n^2}{k} \end{aligned} \tag{12}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\} Y(s) - \left\{s + 2\xi\omega_n\right\} y(0) - y'(0) = \frac{\omega_n^2}{k} \tag{13}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\} Y(s) = \frac{\omega_n^2}{k} + \left\{s + 2\xi\omega_n\right\} y(0) + y'(0) \tag{14}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 - (\xi\omega_n)^2 + \omega_n^2 \tag{15}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_n^2 \left(1 - \xi^2\right) \tag{16}$$

Let

$$\omega_d = \omega_n \sqrt{1 - \xi^2} \tag{17}$$

Substitute equation (17) into (16).

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_d^2 \tag{18}$$

$$\left\{(s + \xi\omega_n)^2 + \omega_d^2\right\} Y(s) = \frac{\omega_n^2}{k} + \{s + 2\xi\omega_n\} y(0) + y'(0) \tag{19}$$

$$Y(s) = \frac{\omega_n^2}{k} \left\{ \frac{1}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} + \left\{ \frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} y(0) + \left\{ \frac{1}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} y'(0) \tag{20}$$

Assume that the initial conditions are zero.

$$Y(s) = \frac{\omega_n^2}{k} \left\{ \frac{1}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \tag{21}$$

Take the inverse Laplace transform.

$$y(t) = \frac{\omega_n^2}{k\,\omega_d} \exp(-\xi\omega_n t) \sin(\omega_d t) \quad (22)$$

$$y(t) = \frac{1}{m\,\omega_d} \exp(-\xi\omega_n t) \sin(\omega_d t) \quad (23)$$

The impulse response function h(t) is

$$h(t) = \frac{1}{m\,\omega_d} \exp(-\xi\omega_n t) \sin(\omega_d t) \quad , \quad t > 0 \quad (24)$$

Consider a system subjected to a force which varies arbitrarily with the time. The response of the mass can be calculated using the impulse response function embedded in a convolution integral.

Reference

1. T. Irvine, The Steady-State Response of a Single-degree-of-freedom System Subjected to a Harmonic Force, Revision A, 2000.
2. D.E. Newland, An Introduction to Random Vibrations, Spectral & Wavelet Analysis, Third Edition, New York, 2005.
3. T. Irvine, Partial Fractions in Shock and Vibration Analysis, Rev F, Vibrationdata, 2010.

APPENDIX A

The impulse response function can also be found from the steady-state transfer function. The transfer function $H(\omega)$ for the system in Figure 1 was derived in Reference 1.

$$H(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{{\omega_n}^2}{k}\left[\frac{-1}{\omega^2 - j\omega\left(2\xi\omega_n\right) - {\omega_n}^2}\right] \tag{A-1}$$

The denominator on the right hand side can be factored such that

$$H(\omega) = -\frac{{\omega_n}^2}{k}\left[\frac{1}{\left(\omega - j\xi\omega_n + \omega_d\right)\left(\omega - j\xi\omega_n - \omega_d\right)}\right] \tag{A-2}$$

Partial fraction expansion yields

$$H(\omega) = \left\{\frac{{\omega_n}^2}{2\,k\,\omega_d}\right\}\left\{\frac{1}{\omega - j\xi\omega_n + \omega_d} - \frac{1}{\omega - j\xi\omega_n - \omega_d}\right\} \tag{A-3}$$

Reference 2 shows that impulse response function is the inverse Fourier transform of the transfer function.

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\exp\left(j\omega t\right)d\omega \tag{A-4}$$

Further note that $t > 0$.

$$h(t) = \frac{1}{2\pi}\left\{\frac{\omega_n^{\,2}}{2k\,\omega_d}\right\}\int_{-\infty}^{\infty}\left\{\frac{1}{\omega - j\xi\omega_n + \omega_d} - \frac{1}{\omega - j\xi\omega_n - \omega_d}\right\}\exp(j\omega t)\,d\omega \qquad \text{(A-5)}$$

$$h(t) = \frac{1}{2\pi}\left\{\frac{\omega_n^{\,2}}{2k\,\omega_d}\right\}\int_{-\infty}^{\infty}\left\{\frac{1}{\omega - j\xi\omega_n + \omega_d}\right\}\exp(j\omega t)\,d\omega$$

$$-\frac{1}{2\pi}\left\{\frac{\omega_n^{\,2}}{2k\,\omega_d}\right\}\int_{-\infty}^{\infty}\left\{\frac{1}{\omega - j\xi\omega_n - \omega_d}\right\}\exp(j\omega t)\,d\omega$$

(A-6)

Substitute variables as follows

$$u = \omega - j\xi\omega_n + \omega_d \qquad \text{(A-7)}$$

$$\omega = u + j\xi\omega_n - \omega_d \qquad \text{(A-8)}$$

$$du = d\omega \qquad \text{(A-9)}$$

$$v = \omega - j\xi\omega_n - \omega_d \qquad \text{(A-10)}$$

$$\omega = v + j\xi\omega_n + \omega_d \qquad \text{(A-11)}$$

$$dv = d\omega \qquad \text{(A-12)}$$

$$h(t) = \frac{1}{2\pi}\left\{\frac{\omega_n^{\;2}}{2k\,\omega_d}\right\}\int_{-\infty}^{\infty}\left\{\frac{1}{u}\right\}\exp\left(j\left[u + j\xi\omega_n - \omega_d\right]t\right)du$$

$$-\frac{1}{2\pi}\left\{\frac{\omega_n^{\;2}}{2k\,\omega_d}\right\}\int_{-\infty}^{\infty}\left\{\frac{1}{v}\right\}\exp\left(j\left[v + j\xi\omega_n + \omega_d\right]t\right)dv$$

(A-13)

$$h(t) = \frac{1}{2\pi}\left\{\frac{\omega_n^{\;2}}{2k\,\omega_d}\right\}\int_{-\infty}^{\infty}\left\{\frac{1}{u}\right\}\exp\left(\left[ju - \xi\omega_n - j\omega_d\right]t\right)du$$

$$-\frac{1}{2\pi}\left\{\frac{\omega_n^{\;2}}{2k\,\omega_d}\right\}\int_{-\infty}^{\infty}\left\{\frac{1}{v}\right\}\exp\left(\left[jv - \xi\omega_n + j\omega_d\right]t\right)dv$$

(A-14)

$$h(t) = \frac{1}{2\pi}\left\{\frac{\omega_n^{\;2}}{2k\,\omega_d}\right\}\int_{-\infty}^{\infty}\left\{\frac{1}{u}\right\}\exp(jut)\exp\left(-\xi\omega_n t\right)\exp\left(-j\omega_d t\right)du$$

$$-\frac{1}{2\pi}\left\{\frac{\omega_n^{\;2}}{2k\,\omega_d}\right\}\int_{-\infty}^{\infty}\left\{\frac{1}{v}\right\}\exp(jvt)\exp\left(-\xi\omega_n t\right)\exp\left(+j\omega_d t\right)dv$$

(A-15)

$$h(t) = \frac{1}{2\pi}\exp\left(-\xi\omega_n t\right)\exp\left(-j\omega_d t\right)\left\{\frac{\omega_n^{\,2}}{2k\,\omega_d}\right\}\int_{-\infty}^{\infty}\left\{\frac{1}{u}\right\}\exp\left(jut\right)du$$

$$-\frac{1}{2\pi}\exp\left(-\xi\omega_n t\right)\exp\left(+j\omega_d t\right)\left\{\frac{\omega_n^{\,2}}{2k\,\omega_d}\right\}\int_{-\infty}^{\infty}\left\{\frac{1}{v}\right\}\exp\left(jvt\right)dv$$

(A-16)

Given that limits go to infinity, the two integral terms are the same.

$$h(t) = \frac{1}{2\pi}\exp\left(-\xi\omega_n t\right)\exp\left(-j\omega_d t\right)\left\{\frac{\omega_n^{\,2}}{2k\,\omega_d}\right\}\int_{-\infty}^{\infty}\left\{\frac{1}{u}\right\}\exp\left(jut\right)du$$

$$-\frac{1}{2\pi}\exp\left(-\xi\omega_n t\right)\exp\left(+j\omega_d t\right)\left\{\frac{\omega_n^{\,2}}{2k\,\omega_d}\right\}\int_{-\infty}^{\infty}\left\{\frac{1}{u}\right\}\exp\left(jut\right)du$$

(A-17)

$$h(t) = \frac{1}{2\pi}\exp\left(-\xi\omega_n t\right)\left\{\exp\left(-j\omega_d t\right)-\exp\left(+j\omega_d t\right)\right\}\left\{\frac{\omega_n^{\,2}}{2k\,\omega_d}\right\}\int_{-\infty}^{\infty}\left\{\frac{1}{u}\right\}\exp\left(jut\right)du$$

(A-18)

$$h(t) = -j\,\frac{1}{\pi}\left\{\frac{\omega_n^{\,2}}{k\,\omega_d}\right\}\exp\left(-\xi\omega_n t\right)\sin\left(\omega_d t\right)\int_{-\infty}^{\infty}\left\{\frac{1}{u}\right\}\exp\left(jut\right)du$$

(A-19)

$$h(t) = -j\,\frac{1}{\pi}\left\{\frac{1}{m\,\omega_d}\right\}\exp\left(-\xi\omega_n t\right)\sin\left(\omega_d t\right)\int_{-\infty}^{\infty}\left\{\frac{1}{u}\right\}\exp\left(jut\right)du$$

(A-20)

$$h(t) = -j\,\frac{1}{\pi}\left\{\frac{1}{m\,\omega_d}\right\}\exp\left(-\xi\omega_n t\right)\sin\left(\omega_d t\right)\int_{-\infty}^{\infty}\left\{\frac{1}{u}\right\}\{\cos(ut) + j\sin(ut)\}du$$

(A-21)

Again note that t > 0.

$$h(t) = -j\,\frac{2}{\pi}\left\{\frac{1}{m\,\omega_d}\right\}\exp\left(-\xi\omega_n t\right)\sin\left(\omega_d t\right)\int_{0^+}^{\infty}\left\{\frac{1}{u}\right\}\{\cos(ut) + j\sin(ut)\}du$$

(A-22)

$$h(t) = \frac{2}{\pi}\left\{\frac{1}{m\,\omega_d}\right\}\exp\left(-\xi\omega_n t\right)\sin\left(\omega_d t\right)\int_{0^+}^{\infty}\left\{\frac{1}{u}\right\}\{-j\cos(ut) + \sin(ut)\}du$$

(A-23)

The impulse response function must be real.

$$h(t) = \frac{2}{\pi}\left\{\frac{1}{m\,\omega_d}\right\}\exp\left(-\xi\omega_n t\right)\sin\left(\omega_d t\right)\int_{0^+}^{\infty}\left\{\frac{1}{u}\sin(ut)\right\}du \tag{A-24}$$

From integral tables,

$$\int_{0^+}^{\infty}\left\{\frac{1}{u}\sin(ut)\right\}du = \frac{\pi}{2} \tag{A-25}$$

The impulse response function is thus

$$h(t) = \left\{ \frac{1}{m\,\omega_d} \right\} \exp\left(-\xi\omega_n t \right) \sin\left(\omega_d t \right) \tag{A-26}$$