THE IMPULSE RESPONSE FUNCTION FOR BASE EXCITATION

By Tom Irvine
Email: tomirvine@aol.com

February 10, 2012

Introduction

Consider the single-degree-of-freedom system in Figure 1.

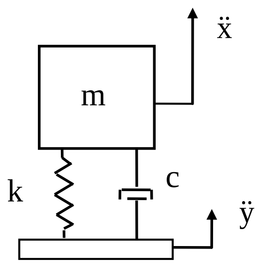

Figure 1.

where

m	=	Mass
c	=	viscous damping coefficient
k	=	Stiffness
x	=	absolute displacement of the mass
y	=	base input displacement

A free-body diagram is shown in Figure 2.

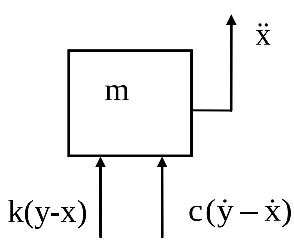

Figure 2.

Summation of forces in the vertical direction

$$\sum F = m\ddot{x} \tag{1}$$

$$m\ddot{x} = c(\dot{y} - \dot{x}) + k(y - x) \tag{2}$$

$$\begin{aligned} \text{Let } z &= x - y \quad \text{(relative displacement)} \\ \dot{z} &= \dot{x} - \dot{y} \\ \ddot{z} &= \ddot{x} - \ddot{y} \\ \ddot{x} &= \ddot{z} + \ddot{y} \end{aligned}$$

Substituting the relative displacement terms into equation (2) yields

$$m(\ddot{z} + \ddot{y}) = -c\dot{z} - kz \tag{3}$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{4}$$

Dividing through by mass yields

$$\ddot{z} + (c/m)\dot{z} + (k/m)z = -\ddot{y} \tag{5}$$

By convention,

$$(c/m) = 2\xi\omega_n \tag{6}$$

$$(k/m) = \omega_n^2 \tag{7}$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

Substitute the convention terms into equation (5).

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^2 z = -\ddot{y} \tag{8}$$

Let $\hat{I}$ = total impulse

$$y(t) = -\hat{I}\,\delta(t) \tag{9}$$

.

$$\ddot{z} + 2\xi\omega_n \dot{z} + \omega_n^2 z = -\hat{I}\,\delta(t) \quad (10)$$

$$\begin{aligned} & s^2 Z(s) - sz(0) - \dot{z}(0) \\ & + 2\xi\omega_n sZ(s) - 2\xi\omega_n z(0) \\ & + \omega_n^2 Z(s) = \qquad \hat{I} \end{aligned} \quad (11)$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\} Z(s) - \left\{s + 2\xi\omega_n\right\} z(0) - \dot{z}(0) = \quad -\hat{I} \quad (12)$$

$$\left\{(s + \xi\omega_n)^2 + \omega_d^2\right\} Z(s) = \quad -\hat{I} + \left\{s + 2\xi\omega_n\right\} z(0) + \dot{z}(0) \quad (13)$$

$$Z(s) =$$

$$\begin{aligned} & -\hat{I}\left\{\frac{1}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} \\ & + \left\{\frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} z(0) + \left\{\frac{1}{(s + \xi\omega_n)^2 + \omega_d^2}\right\} \dot{z}(0) \end{aligned} \quad (14)$$

Relative Displacement

The relative displacement is

$$z(t) = z(0)\exp(-\xi\omega_n t)\left\{\cos(\omega_d t) + \left[\frac{\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\}$$

$$+\dot{z}(0)\left[\frac{1}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t)$$

$$-\hat{I}\left[\frac{1}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t)$$

(15)

The relative displacement for zero initial conditions is

$$z(t) = -\hat{I}\left[\frac{1}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t) \quad (16)$$

The impulse response function is

$$h_d(t) = -\frac{1}{\omega_d}\left[\exp(-\xi\omega_n t)\right]\left[\sin\omega_d t\right] \quad (17)$$

The corresponding Laplace transform is

$$H_d(s) = -\left[\frac{1}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right] = -\left[\frac{1}{(s+\xi\omega_n)^2 + \omega_d{}^2}\right] \quad (18)$$

Relative Velocity

The relative velocity is

$$
\begin{aligned}
\dot{z}(t) = & -\xi\omega_n z(0)\exp(-\xi\omega_n t)\left\{\cos(\omega_d t) + \left[\frac{\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\} \\
& + z(0)\exp(-\xi\omega_n t)\{-\omega_d \sin(\omega_d t) + \xi\omega_n \cos(\omega_d t)\} \\
& - \dot{z}(0)\left[\frac{\xi\omega_n}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t) \\
& + \dot{z}(0)\exp(-\xi\omega_n t)\cos(\omega_d t) \\
& + \hat{I}\left[\frac{\xi\omega_n}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t) \\
& - \hat{I}\exp(-\xi\omega_n t)\cos(\omega_d t)
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\dot{z}(t) = & - z(0)\exp(-\xi\omega_n t)\left[\frac{\xi^2\omega_n^{\,2}}{\omega_d} + \omega_d\right]\sin(\omega_d t) \\
& + \dot{z}(0)\exp(-\xi\omega_n t)\left[\cos(\omega_d t) - \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right] \\
& - \hat{I}\exp(-\xi\omega_n t)\left[\cos(\omega_d t) - \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right]
\end{aligned}
\tag{20}
$$

$$\dot{z}(t) = -z(0)\exp(-\xi\omega_n t)\left[\frac{\xi^2\omega_n{}^2 + \omega_d{}^2}{\omega_d{}^2}\right]\sin(\omega_d t)$$

$$+\dot{z}(0)\exp(-\xi\omega_n t)\left[\cos(\omega_d t) - \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right]$$

$$-\hat{\mathrm{I}}\exp(-\xi\omega_n t)\left[\cos(\omega_d t) - \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right] \tag{21}$$

The relative velocity for zero initial conditions is

$$\dot{z}(t) = -\hat{\mathrm{I}}\ \exp(-\xi\omega_n t)\left[\cos(\omega_d t) - \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right] \tag{22}$$

The impulse response function is

$$h_v(t) = -\hat{\mathrm{I}}\ \exp(-\xi\omega_n t)\left[\cos(\omega_d t) - \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right] \tag{23}$$

The corresponding Laplace transform is

$$H_v(s) = -\left[\frac{s}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right] = -\left[\frac{s}{(s+\xi\omega_n)^2 + \omega_d{}^2}\right] \tag{24}$$

Relative Acceleration

The relative acceleration is

$$\ddot{z}(t) = -\hat{I}\,\delta(t) - 2\xi\omega_n \dot{z}(t) - \omega_n{}^2 z(t) \tag{25}$$

$$\ddot{z}(t) =$$
$$\hat{I}\left\{ -\delta(t) + 2\xi\omega_n \exp(-\xi\omega_n t)\left[\cos(\omega_d t) - \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right] + \omega_n{}^2\left[\frac{1}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t)\right\} \tag{26}$$

$$\ddot{z}(t) = \hat{I}\left\{ -\delta(t) + \exp(-\xi\omega_n t)\left[2\xi\omega_n \cos(\omega_d t) + \frac{\omega_n{}^2 - 2(\xi\omega_n)^2}{\omega_d}\sin(\omega_d t)\right]\right\} \tag{27}$$

$$\ddot{z}(t) = \hat{I}\left\{ -\delta(t) + \exp(-\xi\omega_n t)\left[2\xi\omega_n \cos(\omega_d t) + \frac{\omega_n{}^2}{\omega_d}\left(1 - 2\xi^2\right)\sin(\omega_d t)\right]\right\} \tag{28}$$

The Laplace transform of the relative acceleration is

$$Z_{ra}(s) = \left\{-1 + \frac{2\xi\omega_n(s + \xi\omega_n) + \omega_n{}^2\left(1 - 2\xi^2\right)}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{29}$$

$$Z_{ra}(s) = \left\{-1 + \frac{2\xi\omega_n s + \omega_n{}^2}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{29}$$

$$Z_{ra}(s) = \left\{ -\frac{s^2 + 2\xi\omega_n s + \omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} + \frac{2\xi\omega_n s + \omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \tag{30}$$

$$Z_{ra}(s) = -\frac{s^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{31}$$

<u>Absolute Acceleration</u>

The absolute acceleration is

$$\ddot{x}(t) = \ddot{z}(t) + \hat{I}\,\delta(t) \tag{32}$$

$$\ddot{x}(t) = -2\xi\omega_n \dot{z}(t) - \omega_n^2 z(t) \tag{33}$$

$$\ddot{x}(t) = 2\xi\omega_n \hat{I}\, \exp(-\xi\omega_n t)\left[\cos(\omega_d t) - \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right] + \omega_n^2 \hat{I} \left[\frac{1}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t) \tag{34}$$

$$\ddot{x}(t) = \omega_n \hat{I}\left\{2\xi\, \exp(-\xi\omega_n t)\left[\cos(\omega_d t) - \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right] + \left[\frac{\omega_n}{\omega_d}\right]\exp(-\xi\omega_n t)\sin(\omega_d t)\right\} \tag{35}$$

$$\ddot{x}(t) = \omega_n \hat{I}\left\{ \exp(-\xi\omega_n t)\left[2\xi\cos(\omega_d t) + \left(\frac{\omega_n}{\omega_d} - \frac{2\xi^2\omega_n}{\omega_d}\right)\sin(\omega_d t)\right]\right\} \tag{36}$$

$$\ddot{x}(t) = \omega_n \hat{\mathrm{I}}\left\{ \exp\left(-\xi\omega_n t\right)\left[2\xi\cos\left(\omega_d t\right) + \frac{\omega_n}{\omega_d}\left(1-2\xi^2\right)\sin\left(\omega_d t\right)\right]\right\} \tag{37}$$

$$\ddot{x}(t) = \hat{\mathrm{I}}\left\{ \exp\left(-\xi\omega_n t\right)\left[2\xi\omega_n\cos\left(\omega_d t\right) + \frac{\omega_n^{\,2}}{\omega_d}\left(1-2\xi^2\right)\sin\left(\omega_d t\right)\right]\right\} \tag{38}$$

The Laplace transform of the absolute acceleration is

$$X_a(s) = \left[\frac{2\xi\omega_n s + 2\left(\xi\omega_n\right)^2 + \omega_n^{\,2}\left(1-2\xi^2\right)}{s^2 + 2\xi\omega_n s + \omega_n^{\,2}}\right] \tag{39}$$

$$X_a(s) = \left[\frac{2\xi\omega_n s + \omega_n^{\,2}}{s^2 + 2\xi\omega_n s + \omega_n^{\,2}}\right] \tag{40}$$

<u>References</u>

1. T. Irvine, Table of Laplace Transforms, Revision J, Vibrationdata, 2011.

2. T. Irvine, The Steady-state Response of Single-degree-of-freedom System to a Harmonic Base Excitation, Vibrationdata, 2004.