BENDING ANALYSIS OF STATICALLY INDETERMINANT BEAMS
Revision A

By Tom Irvine
Email: tomirvine@aol.com

July 18, 1999

Clamped-Pinned-Free Beam

Consider the uniform beam in Figure A-1. The end mass represents an applied load. Assume that the mass of the beam itself is negligible.

A free body diagram is shown in Figure A-2.

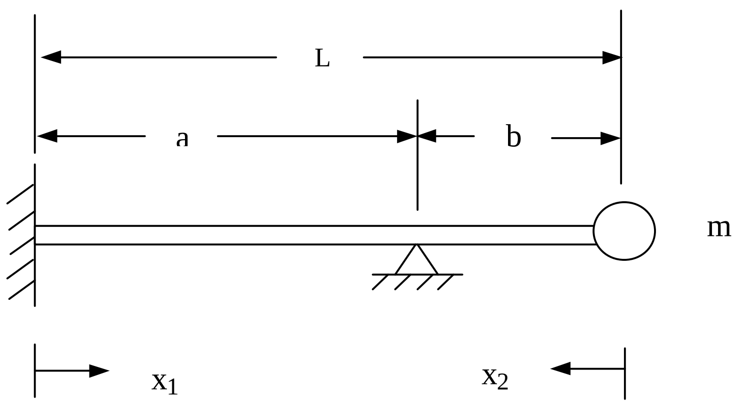

Figure A-1.

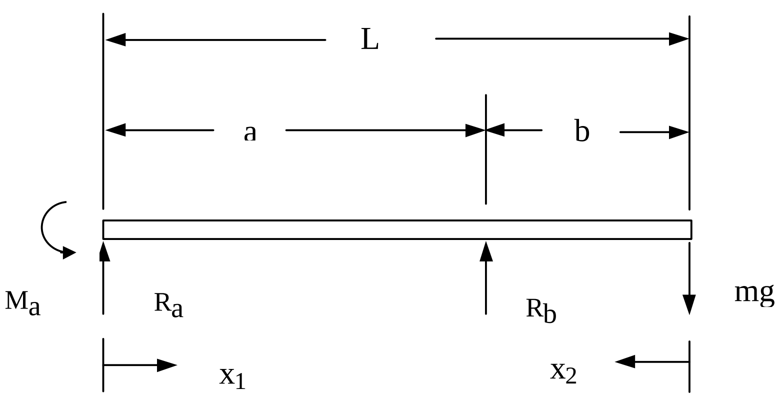

Figure A-2.

The reaction forces are given by R_a and R_b. The reaction moment is M_a.

Determine the following:

1. The reaction forces and reaction moment.
2. The displacement function and its derivatives for each span.
3. The moment relationship at the intermediate, pinned boundary.
4. The effective stiffness at the free end.

Apply Newton's law for static equilibrium.

$$+\uparrow \sum \text{forces} = 0 \tag{A-1}$$

$$R_a + R_b - mg = 0 \tag{A-2}$$

At the left boundary,

$$\circlearrowright + \sum \text{moments} = M_a \tag{A-3}$$

$$M_a + R_b\, a - mg\, L = 0 \tag{A-4}$$

Now consider a segment of the beam, starting from the left boundary as shown in Figure A-3.

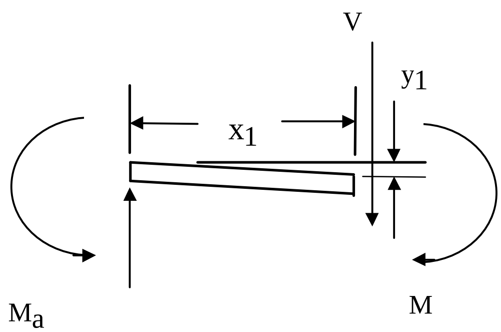

Figure A-3.

V is the shear force.
M is the bending moment.
y_1 is the displacement at position x_1.

Note that a positive displacement is downward.

Sum the moments at the right side of the segment.

$$\circlearrowright + \sum \text{moments} = 0 \tag{A-5}$$

$$M_a - R_a x_1 - M = 0 \tag{A-6}$$

$$M = M_a - R_a x_1 \tag{A-7}$$

The moment M and the deflection y are related by the equation

$$M = EIy_1'' \tag{A-8}$$

where

E is the modulus of elasticity.
I is the area moment of inertia.

Substitute equation (A-7) into (A-8).

$$EIy_1'' = M_a - R_a x_1 \tag{A-9}$$

$$y_1'' = \frac{M_a - R_a x_1}{EI} \tag{A-10}$$

Integrate equation (A-10). A constant c_{11} is obtained.

$$y_1' = \frac{M_a x_1 - \frac{1}{2} R_a x_1^2}{EI} + c_{11} \tag{A-11}$$

$$y_1' = \frac{2M_a x_1 - R_a x_1^2}{2EI} + c_{11} \tag{A-12}$$

Integrate equation (A-12). A constant c_{12} is obtained.

$$y_1 = \frac{M_a x_1^2 - \frac{1}{3} R_a x_1^3}{2EI} + c_{11} x + c_{12} \tag{A-13}$$

$$y_1 = \frac{3M_a x_1^2 - R_a x_1^3}{6EI} + c_{11} x + c_{12} \tag{A-14}$$

Consider the zero displacement boundary condition at the fixed end.

$$y_1(0) = 0 \tag{A-15}$$

Apply the boundary condition to equation (A-14).

$$c_{12} = 0 \tag{A-16}$$

Consider the zero slope boundary condition at the fixed end.

$$y_1'(0) = 0 \tag{A-17}$$

Apply the boundary condition to equation (A-12).

$$c_{11} = 0 \tag{A-18}$$

Substitute equations (A-16) and (A-18) into (A-14). The displacement equation becomes

$$y_1 = \frac{3M_a x_1^2 - R_a x_1^3}{6EI} \tag{A-19}$$

The displacement at the pinned location is

$$y_1(a) = \frac{3M_a a^2 - R_a a^3}{6EI} \tag{A-20}$$

The zero displacement boundary condition at the pinned location is

$$y_1(a) = 0 \tag{A-21}$$

Substitute equation (A-19) into (A-21).

$$\frac{3M_a a^2 - R_a a^3}{6EI} = 0 \tag{A-22}$$

$$3M_a - R_a a = 0 \tag{A-23}$$

The bending moment at the fixed boundary can thus be expressed as

$$M_a = \frac{R_a a}{3} \tag{A-24}$$

Substitute equation (A-24) into (A-4).

$$\frac{R_a a}{3} + R_b\ a - mg\ L = 0 \tag{A-25}$$

$$R_a = -3R_b + mg\,\frac{3L}{a} \tag{A-26}$$

Recall equation (A-2), restated as (A-27).

$$R_a + R_b - mg = 0 \tag{A-27}$$

Substitute equation (A-26) into (A-27).

$$-3R_b + mg\,\frac{3L}{a} + Rb - mg = 0 \tag{A-28}$$

Simplify the equation by collecting terms.

$$-2R_b + mg\left[\frac{3L}{a} - 1\right] = 0 \tag{A-29}$$

$$R_b = \frac{mg}{2}\left[\frac{3L}{a} - 1\right] \tag{A-30}$$

$$R_b = \frac{mg}{2a}\left[3L - a\right] \tag{A-31}$$

Equation (A-27) requires

$$R_a = mg - R_b \tag{A-32}$$

Substitute equation (A-31) into (A-32).

$$R_a = mg - \frac{mg}{2a}[3L - a] \tag{A-33}$$

$$R_a = mg\left\{1 - \frac{1}{2a}[3L - a]\right\} \tag{A-34}$$

$$R_a = \frac{mg}{2a}\{2a - [3L - a]\} \tag{A-35}$$

$$R_a = \frac{mg}{2a}\{3a - 3L\} \tag{A-36}$$

$$R_a = \frac{3mg}{2a}\{a - L\} \tag{A-37}$$

Recall that a + b = L. Thus

$$R_a = \frac{-3mg\,b}{2a} \tag{A-38}$$

Recall equation (A-24), restated as equation (A-39).

$$M_a = \frac{R_a a}{3} \tag{A-39}$$

$$M_a = \frac{a}{3}\frac{3mg}{2a}\{a - L\} \tag{A-40}$$

$$M_a = \frac{mg}{2}\{a - L\} \tag{A-41}$$

$$M_a = \frac{-mgb}{2} \tag{A-42}$$

Summarize the reaction forces and the reaction moment.

$$R_a = \frac{-3mg\,b}{2a} \tag{A-43}$$

$$R_b = \frac{mg}{2a}[3L - a] \tag{A-44}$$

$$M_a = \frac{-mgb}{2} \tag{A-45}$$

Substitute equation (A-45) and (A-43) into (A-46).

$$y_1 = \frac{-3\frac{mgb}{2}x_1^2 + \frac{3mgb}{2a}x_1^3}{6EI} \tag{A-46}$$

$$y_1 = \left[\frac{mgb}{6EI}\right]\left[-\frac{3}{2}x_1^2 + \frac{3}{2a}x_1^3\right] \tag{A-47}$$

$$y_1 = \left[\frac{mgb}{4EIa}\right]\left[-ax_1^2 + x_1^3\right] \tag{A-48}$$

Calculate the first derivative.

$$y_1' = \left[\frac{mgb}{4EIa}\right]\left[-2ax_1 + 3x_1^2\right] \tag{A-49}$$

Evaluate the first derivative at the intermediate boundary.

$$y_1'(a) = \left[\frac{mgb}{4EIa}\right]\left[-2a^2 + 3a^2\right] \tag{A-50}$$

$$y_1'(a) = \left[\frac{mgab}{4EI}\right] \tag{A-51}$$

Calculate the second derivative.

$$y_1'' = \left[\frac{mgb}{4EIa}\right][-2a + 6x_1] \tag{A-52}$$

Evaluate the second derivative at the intermediate boundary.

$$y_1''(a) = \left[\frac{mgb}{4EIa}\right][4a] \quad \text{(A-53)}$$

$$y_1''(a) = \left[\frac{mgb}{EI}\right] \quad \text{(A-54)}$$

Consider the right segment shown in Figure A-4.

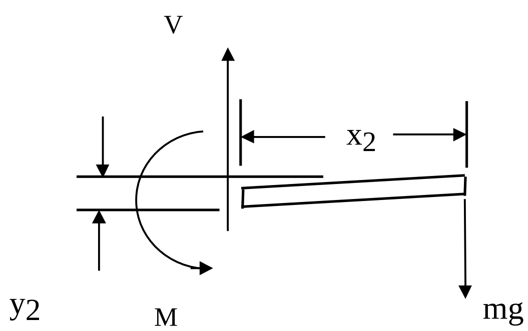

Figure A-4.

Note that a positive displacement is downward.

Sum the moments at the left side of the segment.

$$+ \sum \text{moments} = 0 \quad \text{(A-55)}$$

$$M - mg\,x_2 = 0 \quad \text{(A-56)}$$

$$M = mg\,x_2 \quad \text{(A57)}$$

The moment M and the deflection y are related by the equation

$$M = EIy_2'' \quad \text{(A-58)}$$

$$EIy_2'' = mg\, x_2 \tag{A-59}$$

$$y_2'' = \frac{mg}{EI} x_2 \tag{A-60}$$

Integrate to obtain the slope equation.

$$y_2' = \frac{mg}{2EI} x^2 + c_{21} \tag{A-61a}$$

Evaluate the slope at the intermediate boundary.

$$y_2'(b) = \frac{mg}{2EI} b^2 + c_{21} \tag{A-61b}$$

Integrate to obtain the displacement function.

$$y_2 = \frac{mg}{6EI} x^3 + c_{21}x + c_{22} \tag{A-62}$$

The displacement boundary condition at the intermediate boundary is

$$y_2(b) = 0 \tag{A-63}$$

Substitute equation (A-62) into (A-63).

$$\frac{mg}{6EI} b^3 + c_{21}b + c_{22} = 0 \tag{A-64}$$

$$c_{22} = -\frac{mg}{6EI} b^3 - c_{21}b \tag{A-65}$$

The slope boundary condition at the pinned boundary is

$$y'(a) = -y'(b) \tag{A-66}$$

Substitute equation (A-51) and (A-61b) into (A-66).

$$\left[\frac{mg\,ab}{4\,EI}\right] = -\frac{mg}{2EI} b^2 - c_{21} \tag{A-67}$$

$$c_{21} = -\frac{mg}{2EI} b^2 - \left[\frac{mg\,ab}{4\,EI}\right] \tag{A-68}$$

$$c_{21} = \frac{mg}{EI}\left\{-\frac{1}{2}b^2 - \left[\frac{ab}{4}\right]\right\} \tag{A-69}$$

$$c_{21} = \frac{-mgb}{4EI}\{2b + a\} \tag{A-70}$$

Substitute equation (A-70) into (A-65).

$$c_{22} = -\frac{mg}{6EI}b^3 - \frac{mg}{EI}\left\{-\frac{1}{2}b^2 - \left[\frac{ab}{4}\right]\right\}b \tag{A-71}$$

$$c_{22} = -\frac{mg}{6EI}b^3 - \frac{mg}{EI}\left\{-\frac{1}{2}b^3 - \left[\frac{ab^2}{4}\right]\right\} \tag{A-72}$$

$$c_{22} = \frac{mg}{EI}\left\{-\frac{1}{6}b^3 + \frac{1}{2}b^3 + \frac{ab^2}{4}\right\} \tag{A-73}$$

$$c_{22} = \frac{mg}{EI}\left\{\frac{1}{3}b^3 + \frac{ab^2}{4}\right\} \tag{A-74}$$

Substitute equations (A-74) and (A-70) into (A-62).

$$y_2 = \frac{mg}{6EI}x^3 + \frac{-mgb}{4EI}\{2b + a\}x + \frac{mgb^2}{12EI}\{4b + 3a\} \tag{A-75}$$

$$y_2 = \frac{mg}{EI}\left\{\frac{1}{6}x^3 + \frac{-b}{4}\{2b + a\}x + \frac{b^2}{12}\{4b + 3a\}\right\} \tag{A-76}$$

$$y_2 = \frac{mg}{12EI}\left\{2x^3 - 3b\{2b + a\}x + b^2\{4b + 3a\}\right\} \tag{A-77}$$

The displacement equation is now obtained. The next task is to verify the equation in terms of its boundary conditions.

Evaluate the displacement at the intermediate boundary.

$$y_2(b) = \frac{mg}{12\,EI}\left\{2\,b^3 - 3b\{2b + a\}b + b^2\{4\,b + 3a\}\right\} \tag{A-78}$$

$$y_2(b) = \frac{mg}{12\,EI}\left\{2\,b^3 - 6b^3 - 3ab^2 + 4\,b^3 + 3ab^2\right\} \tag{A-79}$$

$$y_2(b) = 0 \tag{A-80}$$

The zero displacement condition is thus verified. Recalculate the first derivative.

$$y_2' = \frac{mg}{12\,EI}\left\{6\,x^2 - 3b[2b + a]\right\} \tag{A-81}$$

Evaluate the first derivative at the intermediate boundary.

$$y_2'(b) = \frac{mg}{12\,EI}\left\{6\,b^2 - 3b[2b + a]\right\} \tag{A-82}$$

$$y_2'(b) = \frac{mg}{12\,EI}\left\{6\,b^2 - 6b^2 - 3ab\right\} \tag{A-83}$$

$$y_2'(b) = \frac{mg\,ab}{-4\,EI} \tag{A-84}$$

The slope equation (A-66) is thus verified.

Recalculate the second derivative.

$$y_2'' = \frac{mg}{12\,EI}\{12x\} \tag{A-85}$$

$$y_2'' = \frac{mg\,x}{EI} \tag{A-86}$$

Evaluate the second derivative at the intermediate boundary.

$$y_2''(b) = \frac{mg\,b}{EI} \tag{A-87}$$

Recall equation (A-54).

$$y_1''(a) = \left[\frac{mgb}{EI}\right] \qquad \text{(A-88)}$$

Compare equations (A-87) and (A-88).

$$y_1''(a) = y_2''(b) \qquad \text{(A-89)}$$

Equation (A-89) is the bending moment relationship at the intermediate boundary.

Now consider the special case where

$$a = b = \frac{L}{2} \qquad \text{(A-90)}$$

Let

$$Y(x) = \begin{cases} -y_1(x_1) & \text{for } 0 \le x \le a \\ -y_2(x_2) & \text{for } a < x \le L \end{cases} \qquad \text{(A-91)}$$

The negative sign changes the positive displacement to the upward direction.

The resulting displacement function is plotted in Figure A-5.

The displacement at the free end is

$$y_2(0) = \frac{mg}{12\,EI}\left\{b^2\{4\,b + 3a\}\right\} \qquad \text{(A-92)}$$

Hooke’s law for a force F and stiffness k is

$$F = k\,y \qquad \text{(A-93)}$$

$$k = \frac{F}{y} \qquad \text{(A-94)}$$

The applied force at the free end is

$$F = mg \qquad \text{(A-95)}$$

The effective stiffness at the free end is thus

$$k = \frac{mg}{\frac{mg}{12\,EI}\left\{b^2\,[4\,b + 3a]\right\}} \tag{A-96}$$

$$k = \frac{12\,EI}{\left\{b^2\,[4\,b + 3a]\right\}} \tag{A-97}$$

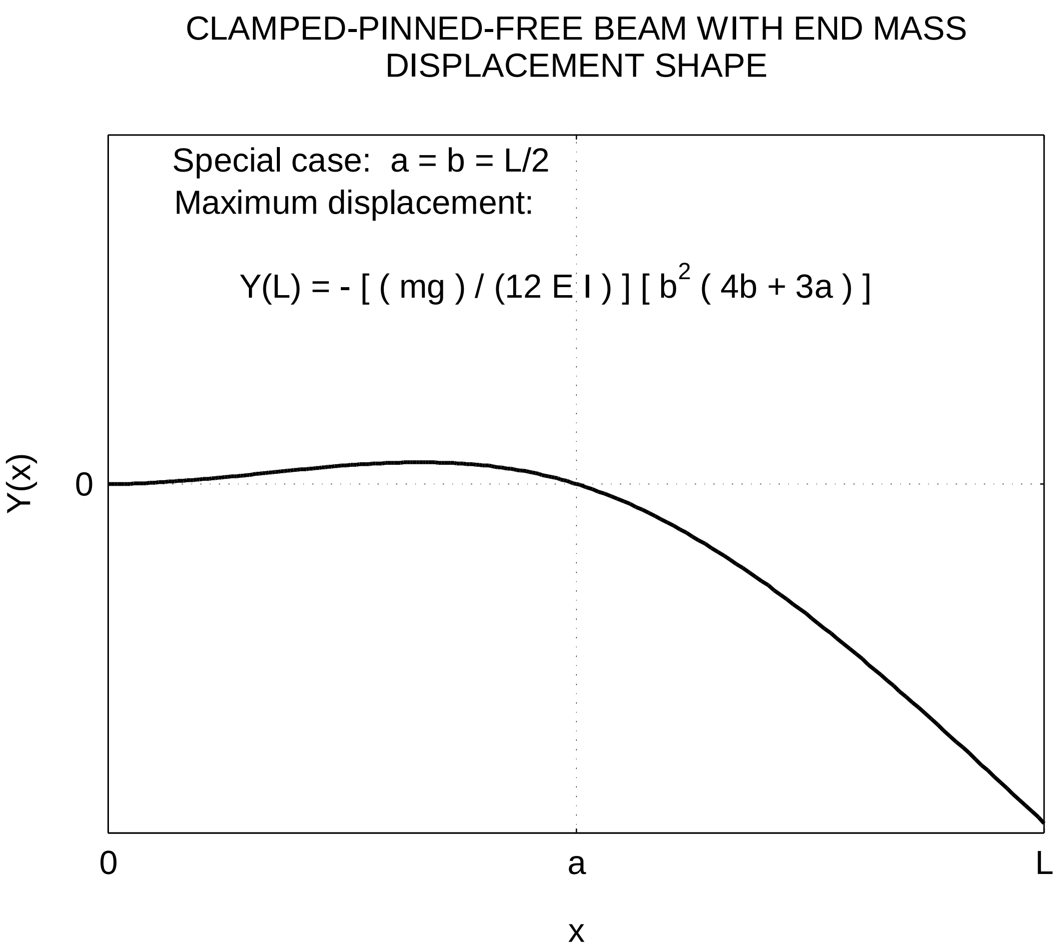

Figure A-5.