A FOUR NODE, ISOPARAMETRIC PLATE BENDING ELEMENT MASS MATRIX
Revision B

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Introduction

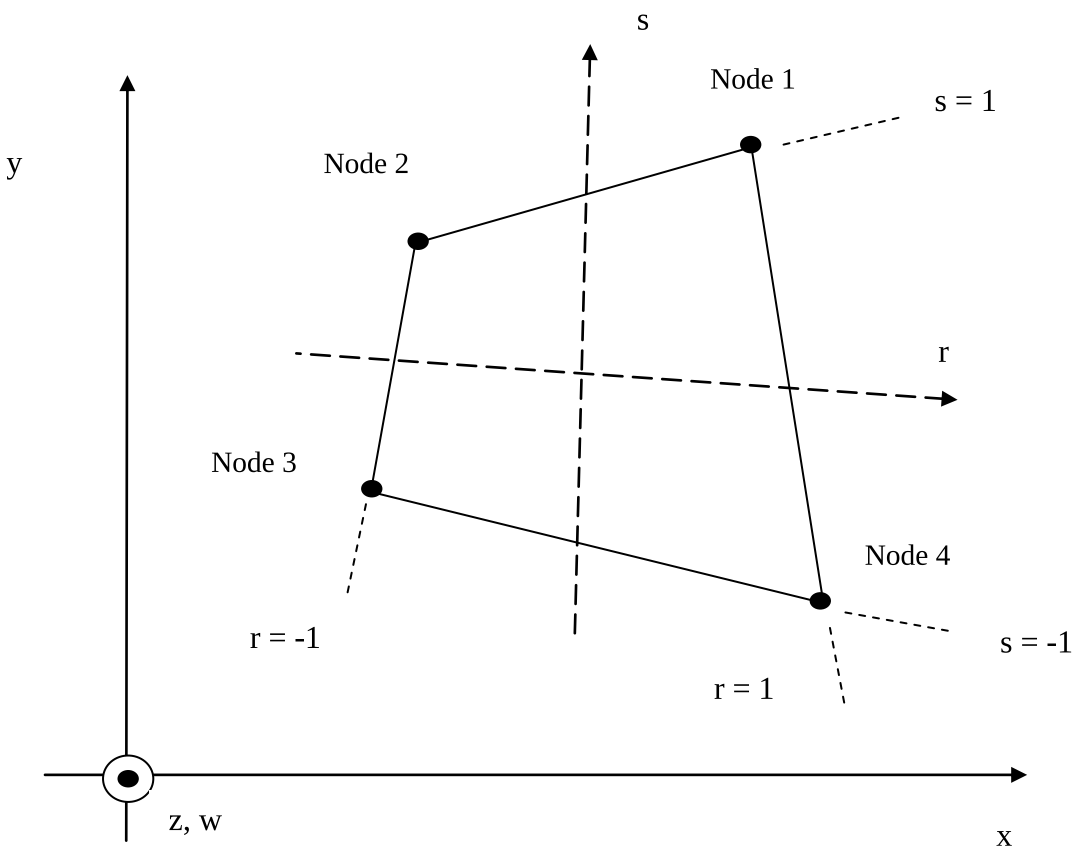

Figure 1.

Note that

$-1 \leq r \leq +1$

$-1 \leq s \leq +1$

Displacement variables:

u	The in-plane displacement along the x-axis
v	The in-plane displacement along the y-axis
w	The out-of-plane displacement along the z-axis
α	The rotation about the x-axis
β	The rotation about the y-axis

The elemental displacement vector is

$$\hat{u} = \begin{bmatrix} w_1 \\ \alpha_1 \\ \beta_1 \\ w_2 \\ \alpha_2 \\ \beta_2 \\ w_3 \\ \alpha_3 \\ \beta_3 \\ w_4 \\ \alpha_4 \\ \beta_4 \end{bmatrix} \tag{1}$$

<u>Interpolation Functions</u>

The coordinate interpolation is

$$x = \frac{1}{4}(1+r)(1+s)x_1 + \frac{1}{4}(1-r)(1+s)x_2 + \frac{1}{4}(1-r)(1-s)x_3 + \frac{1}{4}(1+r)(1-s)x_4 \tag{2}$$

$$y = \frac{1}{4}(1+r)(1+s)y_1 + \frac{1}{4}(1-r)(1+s)y_2 + \frac{1}{4}(1-r)(1-s)y_3 + \frac{1}{4}(1+r)(1-s)y_4 \tag{3}$$

The partial derivatives are

$$\frac{\partial x}{\partial r} = \frac{1}{4}(1+s)x_1 - \frac{1}{4}(1+s)x_2 - \frac{1}{4}(1-s)x_3 + \frac{1}{4}(1-s)x_4 \tag{4}$$

$$\frac{\partial x}{\partial s} = \frac{1}{4}(1+r)x_1 + \frac{1}{4}(1-r)x_2 - \frac{1}{4}(1-r)x_3 - \frac{1}{4}(1+r)x_4 \tag{5}$$

$$\frac{\partial y}{\partial r} = \frac{1}{4}(1+s)y_1 - \frac{1}{4}(1+s)y_2 - \frac{1}{4}(1-s)y_3 + \frac{1}{4}(1-s)y_4 \tag{6}$$

$$\frac{\partial y}{\partial s} = \frac{1}{4}(1+r)x_1 + \frac{1}{4}(1-r)x_2 - \frac{1}{4}(1-r)x_3 - \frac{1}{4}(1+r)x_4 \tag{7}$$

The displacement interpolation is

$$w = \frac{1}{4}(1+r)(1+s)w_1 + \frac{1}{4}(1-r)(1+s)w_2 + \frac{1}{4}(1-r)(1-s)w_3 + \frac{1}{4}(1+r)(1-s)w_4 \tag{8}$$

$$\alpha = \frac{1}{4}(1+r)(1+s)\alpha_1 + \frac{1}{4}(1-r)(1+s)\alpha_2 + \frac{1}{4}(1-r)(1-s)\alpha_3 + \frac{1}{4}(1+r)(1-s)\alpha_4 \tag{9}$$

$$\beta = \frac{1}{4}(1+r)(1+s)\beta_1 + \frac{1}{4}(1-r)(1+s)\beta_2 + \frac{1}{4}(1-r)(1-s)\beta_3 + \frac{1}{4}(1+r)(1-s)\beta_4 \tag{10}$$

Jacobian

The Jacobian matrix J is

$$J = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{bmatrix} \tag{11}$$

By substitution,

J =

$$\begin{bmatrix} [(1+s)x_1 - (1+s)x_2 - (1-s)x_3 + (1-s)x_4]/4 & [(1+s)y_1 - (1+s)y_2 - (1-s)y_3 + (1-s)y_4]/4 \\ [(1+r)x_1 + (1-r)x_2 - (1-r)x_3 - (1+r)x_4]/4 & [(1+r)y_1 + (1-r)y_2 - (1-r)y_3 - (1+r)y_4]/4 \end{bmatrix} \tag{12}$$

The determinant from Reference 2 is

det J =

$$\begin{aligned} &-\left[((s-1)x_3 + (-s-r)x_2 + (r+1)x_1)y_4 + ((1-s)x_4 + (r-1)x_2 + (s-r)x_1)y_3\right]/8 \\ &-\left[((s+r)x_4 + (1-r)x_3 + (-s-1)x_1)y2 + ((-r-1)x_4 + (r-s)x_3 + (s+1)x_2)y1\right]/8 \end{aligned} \tag{13}$$

Mass Element

Assume a constant mass density across the element.

The kinetic energy for a thick plate including rotary inertia is

$$T = \frac{\rho}{2}\int_0^b\int_0^a \left(h\,\dot{w}^2 + \frac{h^3}{12}\dot{\alpha}^2 + \frac{h^3}{12}\dot{\beta}^2 \right) dx\,dy \tag{14}$$

where

α	=	angular rotation about the x-axis
β	=	angular rotation about the y-axis

Equation (14) is taken from Reference 4, equation (6.64).

Divide the kinetic energy into three parts.

$$T = T_1 + T_2 + T_3 \tag{15}$$

$$T_1 = \frac{\rho h}{2}\int_0^b\int_0^a \left(\dot{w}^2\right) dx\,dy \tag{16}$$

$$T_2 = \frac{\rho h^3}{2}\frac{1}{12}\int_0^b\int_0^a \left(\dot{\alpha}^2\right) dx\,dy \tag{17}$$

$$T_3 = \frac{\rho h^3}{2}\frac{1}{12}\int_0^b\int_0^a \left(\dot{\beta}^2\right) dx\,dy \tag{18}$$

Take the mass matrix M as the sum of three parts.

$$M = M_1 + M_2 + M_3 \tag{19}$$

$$M_1 = \rho h \int_{-1}^{1}\int_{-1}^{1} \left\{ P_1^T \; P_1 \right\} \det[J] dr ds \tag{20}$$

$$M_2 = \frac{\rho h^3}{12} \int_{-1}^{1}\int_{-1}^{1} \left\{ P_2^T \; P_2 \right\} \det[J] dr ds \tag{21}$$

$$M_3 = \frac{\rho h^3}{12} \int_{-1}^{1}\int_{-1}^{1} \left\{ P_3^T \; P_3 \right\} \det[J] dr ds \tag{22}$$

Again, the elemental displacement vector is

$$\begin{bmatrix} w \\ \alpha \\ \beta \end{bmatrix} = P_i \begin{bmatrix} w_1 \\ \alpha_1 \\ \beta_1 \\ w_2 \\ \alpha_2 \\ \beta_2 \\ w_3 \\ \alpha_3 \\ \beta_3 \\ w_4 \\ \alpha_4 \\ \beta_4 \end{bmatrix}$$

(23)

The P_1 matrix is

$$P_1 = \frac{1}{4}\begin{bmatrix} (1+r)(1+s) & 0 & 0 & (1-r)(1+s) & 0 & 0 & (1-r)(1-s) & 0 & 0 & (1+r)(1-s) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(24)

The P_2 matrix is

$$P_2 = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1+r)(1+s) & 0 & 0 & (1-r)(1+s) & 0 & 0 & (1-r)(1-s) & 0 & 0 & (1+r)(1-s) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(25)

The P_3 matrix is

$$P_3 = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1+r)(1+s) & 0 & 0 & (1-r)(1+s) & 0 & 0 & (1-r)(1-s) & 0 & 0 & (1+r)(1-s) \end{bmatrix}$$

(26)

Let

$$\hat{a}(r,s) = (1+r)\,(1+s)/4 \tag{27}$$

$$\hat{b}(r,s) = (1-r)\,(1+s)/4 \tag{28}$$

$$\hat{c}(r,s) = (1-r)\,(1-s)/4 \tag{29}$$

$$\hat{d}(r,s) = (1+r)\,(1-s)/4 \tag{30}$$

The P_1 matrix is

$$P_1 = \begin{bmatrix} \hat{a} & 0 & 0 & \hat{b} & 0 & 0 & \hat{c} & 0 & 0 & \hat{d} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(31)

The P_2 matrix is

$$P_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \hat{a} & 0 & 0 & \hat{b} & 0 & 0 & \hat{c} & 0 & 0 & \hat{d} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(32)

The P_3 matrix is

$$P_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{a} & 0 & 0 & \hat{b} & 0 & 0 & \hat{c} & 0 & 0 & \hat{d} \end{bmatrix}$$

(33)

Finite element texts typically recommend evaluation integrals such as equation (21) via Gauss quadrature. But the integrals can also be evaluated symbolically using the software tool wxMaxima.

The result of the symbolic integration is the following mass matrix. The matrix is symmetric and is shown in upper triangular form for brevity.

$$M_1 = \rho h \begin{bmatrix} a^2 & 0 & 0 & ab & 0 & 0 & ac & 0 & 0 & ad & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & b^2 & 0 & 0 & bc & 0 & 0 & bd & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & c^2 & 0 & 0 & cd & 0 & 0 \\ & & & & & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & 0 & 0 & 0 & 0 \\ & & & & & & & & & d^2 & 0 & 0 \\ & & & & & & & & & & 0 & 0 \\ & & & & & & & & & & & 0 \end{bmatrix} \tag{34}$$

$$M_2 = \frac{\rho h^3}{12} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & a^2 & 0 & 0 & ab & 0 & 0 & ac & 0 & 0 & ad & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & b^2 & 0 & 0 & bc & 0 & 0 & bd & 0 \\ & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & c^2 & 0 & 0 & cd & 0 \\ & & & & & & & & 0 & 0 & 0 & 0 \\ & & & & & & & & & 0 & 0 & 0 \\ & & & & & & & & & & d^2 & 0 \\ & & & & & & & & & & & 0 \end{bmatrix} \tag{35}$$

$$M_3 = \frac{\rho h^3}{12}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & a^2 & 0 & 0 & ab & 0 & 0 & ac & 0 & 0 & ad \\ & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & b^2 & 0 & 0 & bc & 0 & 0 & bd \\ & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & c^2 & 0 & 0 & cd \\ & & & & & & & & & 0 & 0 & 0 \\ & & & & & & & & & & 0 & 0 \\ & & & & & & & & & & & d^2 \end{bmatrix} \tag{36}$$

where

$$a^2 = \int_{-1}^{1}\int_{-1}^{1}\{\hat{a}(r,s)\}^2 \det[J(r,s)]drds \tag{37}$$

$$ab = \int_{-1}^{1}\int_{-1}^{1}\{\hat{a}(r,s)\hat{b}(r,s)\}\det[J(r,s)]drds \tag{38}$$

$$ac = \int_{-1}^{1}\int_{-1}^{1}\{\hat{a}(r,s)\hat{c}(r,s)\}\det[J(r,s)]drds \tag{39}$$

$$ad = \int_{-1}^{1}\int_{-1}^{1}\{\hat{a}(r,s)\hat{d}(r,s)\}\det[J(r,s)]drds \tag{40}$$

$$b^2 = \int_{-1}^{1}\int_{-1}^{1}\{\hat{b}(r,s)\}^2 \det[J(r,s)]drds \tag{41}$$

$$bc = \int_{-1}^{1}\int_{-1}^{1}\{\hat{b}(r,s)\hat{c}(r,s)\}\det[J(r,s)]drds \tag{42}$$

$$bd = \int_{-1}^{1}\int_{-1}^{1}\{\hat{b}(r,s)\hat{d}(r,s)\}\det[J(r,s)]drds \tag{43}$$

$$c^2 = \int_{-1}^{1}\int_{-1}^{1}\{\hat{c}(r,s)\}^2 \det[J(r,s)]drds \tag{44}$$

$$cd = \int_{-1}^{1}\int_{-1}^{1}\{\hat{c}(r,s)\hat{d}(r,s)\}\det[J(r,s)]drds \quad (45)$$

$$d^2 = \int_{-1}^{1}\int_{-1}^{1}\{\hat{d}(r,s)\}^2 \det[J(r,s)]drds \quad (46)$$

The coefficients are

$$a^2 = ((x_3 + 2x_2 - 3x_1)y^4 + (x_2 - x_4)y_3 + (-2x_4 - x_3 + 3x_1)y_2 + (3x_4 - 3x_2)y_1)/36 \quad (47)$$

$$ab = ((x_3 + x_2 - 2x_1)y_4 + (-x_4 + 2x_2 - x_1)y_3 + (-x_4 - 2x_3 + 3x_1)y_2 + (2x_4 + x_3 - 3x_2)y_1)/72 \quad (48)$$

$$ac = ((x_3 - x_1)y_4 + (x_2 - x_4)y_3 + (x_1 - x_3)y_2 + (x_4 - x_2)y_1)/72 \quad (49)$$

$$ad = ((2x_3 + x_2 - 3x_1)y_4 + (-2x_4 + x_2 + x_1)y_3 + (-x_4 - x_3 + 2x_1)y_2 + (3x_4 - x_3 - 2x_2)y_1)/72 \quad (50)$$

$$b^2 = ((x_3 - x_1)y_4 + (-x_4 + 3x_2 - 2x_1)y_3 + (3x_1 - 3x_3)y_2 + (x_4 + 2x_3 - 3x_2)y_1)/36 \quad (51)$$

$$bc = ((2x_3 - x_2 - x_1)y_4 + (-2x_4 + 3x_2 - x_1)y_3 + (x_4 - 3x_3 + 2x_1)y_2 + (x_4 + x_3 - 2x_2)y_1)/72 \quad (52)$$

$$bd = ((x_3 - x_1)y_4 + (x_2 - x_4)y_3 + (x_1 - x_3)y_2 + (x_4 - x_2)y_1)/72 \quad (53)$$

$$c^2 = ((3x_3 - 2x_2 - x_1)y_4 + (3x_2 - 3x_4)y_3 + (2x_4 - 3x_3 + x_1)y_2 + (x_4 - x_2)y_1)/36 \tag{54}$$

$$cd = ((3x_3 - x_2 - 2x_1)y_4 + (-3x_4 + 2x_2 + x_1)y_3 + (x_4 - 2x_3 + x_1)y_2 + (2x_4 - x_3 - x_2)y_1)/72 \tag{55}$$

$$d^2 = ((3x_3 - 3x_1)y_4 + (-3x_4 + x_2 + 2x_1)y_3 + (x_1 - x_3)y_2 + (3x_4 - 2x_3 - x_2)y_1)/36 \tag{56}$$

The coefficients are given in Matlab format in Appendix A.

References

1. K. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, Englewood Cliffs, New Jersey, 1982.
2. T. Irvine, Jacobian Matrix and Determinant, Revision C, Vibrationdata, 2012.
3. J. Rao, Dynamics of Plates, Narosa, New Dehli, 1999.
4. M. Petyt, Introduction to Finite Element Vibration Analysis, Second Edition, Cambridge University Press, 2010.

APPENDIX A

Integration Note

Each of the coefficients was calculated using wxMaxima.

The wxMaxima commands for the determinant of the Jacobian as a function of r and s are:

```
j11(r,s):=((1+s)*x1-(1+s)*x2+(-(1-s))*x3+(1-s)*x4)/4;
j12(r,s):=((1+s)*y1-(1+s)*y2+(-(1-s))*y3+(1-s)*y4)/4;
j21(r,s):=((1+r)*x1+(1-r)*x2+(-(1-r))*x3+(-(1+r))*x4)/4;
j22(r,s):=((1+r)*y1+(1-r)*y2+(-(1-r))*y3+(-(1+r))*y4)/4;
detJ(r,s):=ratsimp(determinant(matrix([j11(r,s),j12(r,s)],[j21(r,s),j22
(r,s)])));.
```

The wxMaxima commands for the four internal functions are giving in the following set with the ^ symbol is omitted.

```
a(r,s):=(1+r)*(1+s)/4;
b(r,s):=(1-r)*(1+s)/4;
c(r,s):=(1-r)*(1-s)/4;
d(r,s):=(1+r)*(1-s)/4;
```

Here is an example for term ac in equation (22).

$$ac = \int_{-1}^{1}\int_{-1}^{1}\{\hat{a}(r,s)\hat{c}(r,s)\}\det[J(r,s)]drds \qquad \text{(A-1)}$$

The integration for term ac is:

```
ratsimp(integrate(integrate(a(r,s)*c(r,s)*detJ(r,s),r,-1,1),s,-1,1));

((x3-x1)*y4+(x2-x4)*y3+(x1-x3)*y2+(x4-x2)*y1)/72;
```

Thus

$$ac = ((x_3 - x_1)y_4 + (x_2 - x_4)y_3 + (x_1 - x_3)y_2 + (x_4 - x_2)y_1)/72 \qquad \text{(A-2)}$$

Equation (A-2) is the same as (35) in the main text.

Snippet of Matlab Code for Implementing the Mass Matrix

The user must supply the x and y values at each of the four corners. The final elemental mass matrix must also be multiplied by the (mass/area).

```
%
a2= ((x3+2*x2-3*x1)*y4+(x2-x4)*y3+(-2*x4-x3+3*x1)*y2+(3*x4-3*x2)*y1)/36;
%
ab= ((x3+x2-2*x1)*y4+(-x4+2*x2-x1)*y3+(-x4-2*x3+3*x1)*y2+(2*x4+x3-3*x2)*y1)/72;
%
ac= ((x3-x1)*y4+(x2-x4)*y3+(x1-x3)*y2+(x4-x2)*y1)/72;
%
ad= ((2*x3+x2-3*x1)*y4+(-2*x4+x2+x1)*y3+(-x4-x3+2*x1)*y2+(3*x4-x3-2*x2)*y1)/72;
%
b2= ((x3-x1)*y4+(-x4+3*x2-2*x1)*y3+(3*x1-3*x3)*y2+(x4+2*x3-3*x2)*y1)/36;
%
bc= ((2*x3-x2-x1)*y4+(-2*x4+3*x2-x1)*y3+(x4-3*x3+2*x1)*y2+(x4+x3-2*x2)*y1)/72;
%
bd=((x3-x1)*y4+(x2-x4)*y3+(x1-x3)*y2+(x4-x2)*y1)/72;
%
c2=((3*x3-2*x2-x1)*y4+(3*x2-3*x4)*y3+(2*x4-3*x3+x1)*y2+(x4-x2)*y1)/36;
%
cd=((3*x3-x2-2*x1)*y4+(-3*x4+2*x2+x1)*y3+(x4-2*x3+x1)*y2+(2*x4-x3-x2)*y1)/72;
%
d2=((3*x3-3*x1)*y4+(-3*x4+x2+2*x1)*y3+(x1-x3)*y2+(3*x4-2*x3-x2)*y1)/36;
%
%
q=h^2/12;
%
a2q=a2*q;
%
abq=ab*q;
%
acq=ac*q;
%
adq=ad*q;
%
b2q=b2*q;
%
bcq=bc*q;
%
bdq=bd*q;
%
c2q=c2*q;
%
cdq=cd*q;
%
d2q=d2*q;
%
```

```
mass=[a2,  0,  0, ab,  0,  0, ac,  0,  0, ad,  0,  0;...
       0,a2q,  0,  0,abq,  0,  0,acq,  0,  0,adq,  0;...
       0,  0,a2q,  0,  0,abq,  0,  0,acq,  0,  0,adq;...
      ab,  0,  0, b2,  0,  0, bc,  0,  0, bd,  0,  0;...
       0,abq,  0,  0,b2q,  0,  0,bcq,  0,  0,bdq,  0;...
       0,  0,abq,  0,  0,b2q,  0,  0,bcq,  0,  0,bdq;...
      ac,  0,  0, bc,  0,  0, c2,  0,  0, cd,  0,  0;...
       0,acq,  0,  0,bcq,  0,  0,c2q,  0,  0,cdq,  0;...
       0,  0,acq,  0,  0,bcq,  0,  0,c2q,  0,  0,cdq;...
      ad,  0,  0, bd,  0,  0, cd,  0,  0, d2,  0,  0;...
       0,adq,  0,  0,bdq,  0,  0,cdq,  0,  0,d2q,  0;...
       0,  0,adq,  0,  0,bdq,  0,  0,cdq,  0,  0,d2q];
%
M=mass*rho*h;
M
```