THE EQUATION OF MOTION FOR A MASS-PENDULUM SYSTEM

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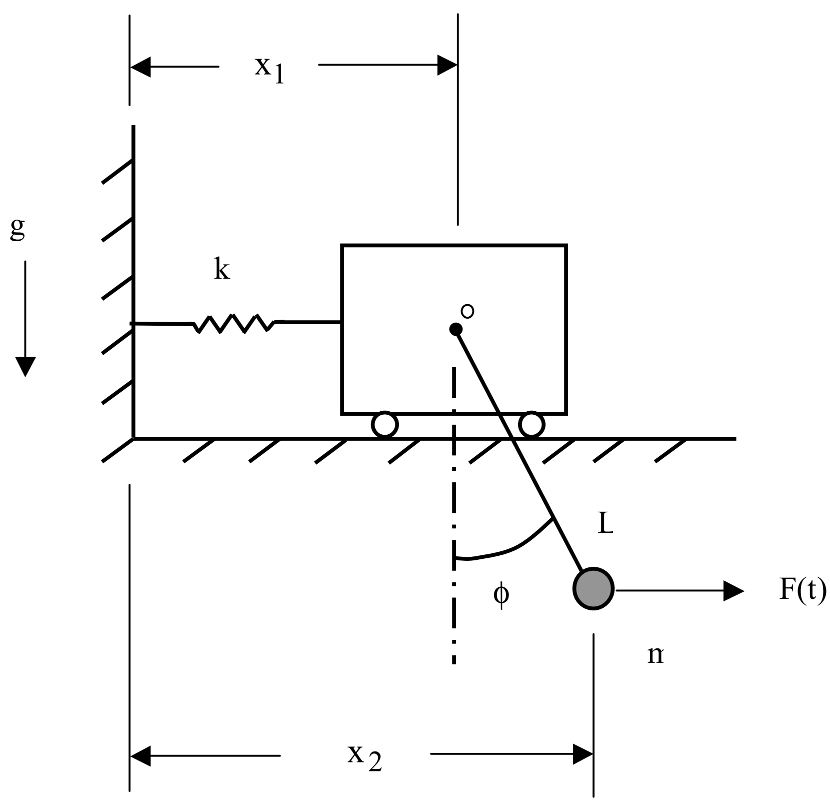
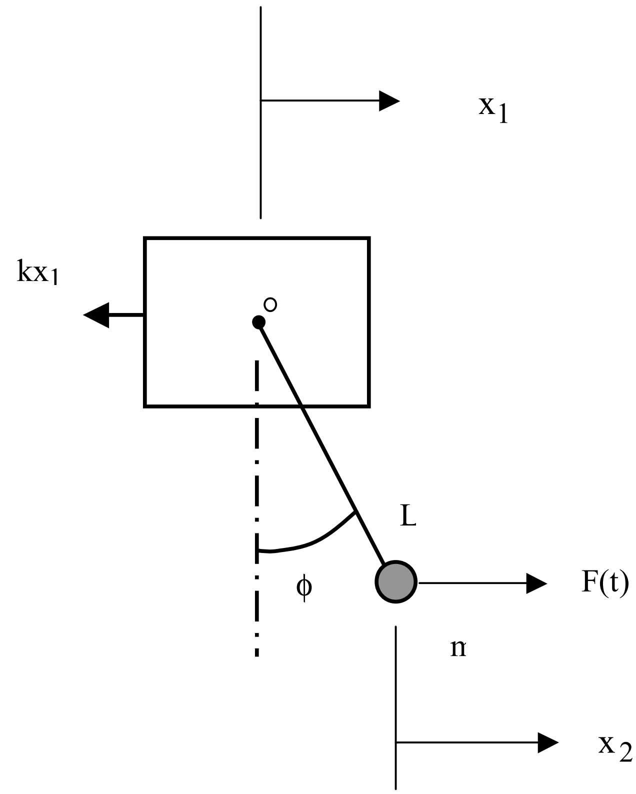
Derive the equations of motion using Newton's law. Assume small angular displacement.





$$\sum F = M\ddot{x}_1 + m\ddot{x}_2 \tag{1}$$

$$M\ddot{x}_1 + m\ddot{x}_2 = -kx_1 + F \tag{2}$$

$$x_2 = x_1 + L\sin\phi \tag{3}$$

$$\dot{x}_2 = \dot{x}_1 + L\cos\phi\dot{\phi} \tag{4}$$

$$\ddot{x}_2 = \ddot{x}_1 + L\cos\phi\ddot{\phi} - L\sin\phi\dot{\phi}^2 \tag{5}$$

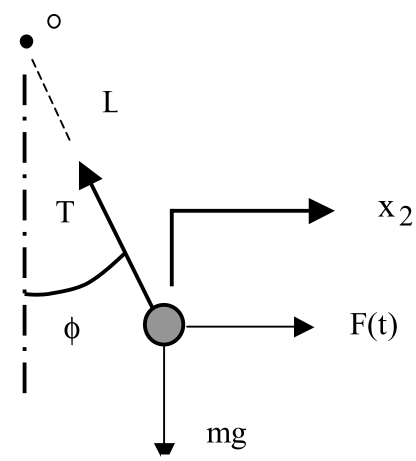
Neglect the higher order term $\sin\phi\dot{\phi}^2$.

For small angular displacement,

$$\ddot{x}_2 \approx \ddot{x}_1 + L\ddot{\phi} \tag{6}$$

$$M\ddot{x}_1 + m[\ddot{x}_1 + L\ddot{\phi}] = -kx_1 + F \tag{7}$$

$$[M+m]\ddot{x}_1 + mL\ddot{\phi} + kx_1 = F \tag{8}$$



The sum of the moments about the pivot point is

$$\sum \hat{M}_o = mL\cos\phi\,\ddot{x}_2 \tag{4}$$

$$mL\cos\phi\,\ddot{x}_2 = -mgL\sin\phi + FL\cos\phi \tag{5}$$

Recall

$$\ddot{x}_2 \approx \ddot{x}_1 + L\ddot{\phi} \tag{6}$$

$$mL\cos\phi[\ddot{x}_1 + L\ddot{\phi}] = -mgL\sin\phi + FL\cos\phi \tag{7}$$

For small angular displacements,

$$mL[\ddot{x}_1 + L\ddot{\phi}] = -mgL\phi + FL \tag{8}$$

$$mL[\ddot{x}_1 + L\ddot{\phi}] + mgL\phi = FL \tag{9}$$

$$m[\ddot{x}_1 + L\ddot{\phi}] + mg\phi = F \tag{10}$$

The equations of motion in matrix form are

$$\begin{bmatrix} M+m & mL \\ m & mL \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{\phi} \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & mg \end{bmatrix} \begin{bmatrix} x_1 \\ \phi \end{bmatrix} = \begin{bmatrix} F(t) \\ F(t) \end{bmatrix} \tag{11}$$

Aside from the forcing function, the result agrees with that for the free vibration problem of a similar system in Reference 1, chapter 2, problems 12 and 23.

Reference

1. W. Seto, Mechanical Vibrations, McGraw-Hill, New York, 1964.

APPENDIX A

Energy Method

Repeat the problem from the main text using the energy method.

$$KE = \frac{1}{2}M\dot{x}_1{}^2 + \frac{1}{2}m\dot{x}_2{}^2 \quad \text{(A-1)}$$

$$KE = \frac{1}{2}M\dot{x}_1{}^2 + \frac{1}{2}m\left[\dot{x}_1 + L\cos\phi\dot{\phi}\right]^2 \quad \text{(A-2)}$$

$$PE = \frac{1}{2}kx_1{}^2 + mg\left[1 - \cos\phi\right]L \quad \text{(A-3)}$$

$$\text{Work} = Fx_2 \quad \text{(A-4)}$$

$$\text{Work} = F[x_1 + L\sin\phi] \quad \text{(A-5)}$$

The energy method is

$$\frac{d}{dt}\left\{KE + PE - \text{Work}\right\} = 0 \quad \text{(A-6)}$$

$$\frac{d}{dt}\left\{\frac{1}{2}M\dot{x}_1{}^2 + \frac{1}{2}m\left[\dot{x}_1 + L\cos\phi\dot{\phi}\right]^2 + \frac{1}{2}kx_1{}^2 + mg\left[1-\cos\phi\right]L - F[x_1 + L\sin\phi]\right\} = 0$$

(A-7)

$$M\dot{x}_1\ddot{x}_1 + m\left[\dot{x}_1 + L\cos\phi\dot{\phi}\right]\left[\ddot{x}_1 - L\sin\phi\dot{\phi}^2 + L\cos\phi\ddot{\phi}\right]$$
$$+ kx_1\dot{x}_1 + mg\sin\phi\dot{\phi}L - F[\dot{x}_1 + L\cos\phi\dot{\phi}] = 0$$

(A-8)

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$$\dot{x}_1\left\{M\ddot{x}_1 + m\left[\ddot{x}_1 - L\sin\phi\dot{\phi}^2 + L\cos\phi\ddot{\phi}\right] + kx_1 - F\right\} = 0$$

(A-9)

$$\dot{\phi}\left\{m[L\cos\phi]\left[\ddot{x}_1 - L\sin\phi\dot{\phi}^2 + L\cos\phi\ddot{\phi}\right] + mg\sin\phi L - F[L\cos\phi]\right\} = 0$$

(A-10)

$$M\ddot{x}_1 + m\left[\ddot{x}_1 - L\sin\phi\dot{\phi}^2 + L\cos\phi\ddot{\phi}\right] + kx_1 - F = 0 \quad \text{(A-11)}$$

$$m[L\cos\phi]\left[\ddot{x}_1 - L\sin\phi\dot{\phi}^2 + L\cos\phi\ddot{\phi}\right] + mg\sin\phi L - F[L\cos\phi] = 0 \quad \text{(A-12)}$$

Apply small angle assumption. Neglect the higher order term $\sin\phi\dot{\phi}^2$.

$$M\ddot{x}_1 + m\left[\ddot{x}_1 + L\ddot{\phi}\right] + kx_1 - F = 0 \tag{A-13}$$

$$[M+m]\ddot{x}_1 + mL\ddot{\phi} + kx_1 - F = 0 \tag{A-14}$$

$$mL\left[\ddot{x}_1 + L\ddot{\phi}\right] + mg\phi L - FL = 0 \tag{A-15}$$

$$m\left[\ddot{x}_1 + L\ddot{\phi}\right] + mg\phi - F = 0 \tag{A-16}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} M+m & mL \\ m & mL \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{\phi} \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & mg \end{bmatrix}\begin{bmatrix} x_1 \\ \phi \end{bmatrix} = \begin{bmatrix} F(t) \\ F(t) \end{bmatrix} \tag{A-17}$$