

MATRIX DEFLATION FOR REMOVING THE RIGID-BODY MODE
FROM THE GENERALIZED EIGENVALUE PROBLEM
Revision B

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The purpose of the tutorial is to build upon the matrix deflation method from Reference 1.

Consider the longitudinal vibration of a free-free rod. Use the finite element method to model the rod with two elements. The solution is taken from Reference 2.

The generalized eigenvalue problem can be expressed as

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \left(\frac{\lambda}{6}\right) \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad (1)$$

Each eigenvalue is represented by λ .

The eigenvector is $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$.

The eigenvector represents the modal displacement.

The eigenvalues for the sample problem are

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 12 \end{bmatrix} \quad (2)$$

The first eigenvalue is zero since the free-free beam has a rigid-body mode.

Now remove the rigid-body mode from the eigenvalue problem.

The eigenvalue problem has the form

$$\underline{\mathbf{K}} \bar{\mathbf{u}} = \lambda \underline{\mathbf{M}} \bar{\mathbf{u}} \quad (3)$$

where

$\underline{\mathbf{K}}$ is the stiffness matrix

$\underline{\mathbf{M}}$ is the mass matrix

For a problem with three degrees-of-freedom,

$$\underline{\mathbf{K}} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \quad (4)$$

$$\underline{\mathbf{M}} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \quad (5)$$

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \lambda \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad (6)$$

Consider the rigid-body mode. The eigenvector is

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (7)$$

Any other natural mode must be orthogonal to the rigid-body mode. Thus

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0 \quad (8)$$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} m_{11}u_1 + m_{12}u_2 + m_{13}u_3 \\ m_{21}u_1 + m_{22}u_2 + m_{23}u_3 \\ m_{31}u_1 + m_{32}u_2 + m_{33}u_3 \end{bmatrix} = 0 \quad (9)$$

$$\begin{aligned} & m_{11}u_1 + m_{12}u_2 + m_{13}u_3 \\ & + m_{21}u_1 + m_{22}u_2 + m_{23}u_3 \\ & + m_{31}u_1 + m_{32}u_2 + m_{33}u_3 = 0 \end{aligned} \quad (10)$$

$$\begin{aligned} & (m_{11} + m_{21} + m_{31})u_1 \\ & + (m_{12} + m_{22} + m_{32})u_2 \\ & + (m_{13} + m_{23} + m_{33})u_3 = 0 \end{aligned} \quad (11)$$

$$u_3 = \frac{(m_{11} + m_{21} + m_{31})}{(m_{13} + m_{23} + m_{33})}u_1 + \frac{(m_{12} + m_{22} + m_{32})}{(m_{13} + m_{23} + m_{33})}u_2 \quad (12)$$

Equation (12) is a constraint equation.

The constraint is applied as follows

$$\begin{aligned}
 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_c &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{(m_{11} + m_{21} + m_{31})}{(m_{13} + m_{23} + m_{33})} & -\frac{(m_{12} + m_{22} + m_{32})}{(m_{13} + m_{23} + m_{33})} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 \\ -\frac{(m_{11} + m_{21} + m_{31})}{(m_{13} + m_{23} + m_{33})} & -\frac{(m_{12} + m_{22} + m_{32})}{(m_{13} + m_{23} + m_{33})} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
 \end{aligned} \tag{13}$$

The transformation matrix is

$$\underline{\underline{C}} = \begin{bmatrix} 1 & 0 \\ -\frac{(m_{11} + m_{21} + m_{31})}{(m_{13} + m_{23} + m_{33})} & -\frac{(m_{12} + m_{22} + m_{32})}{(m_{13} + m_{23} + m_{33})} \end{bmatrix} \tag{14}$$

Transform the eigenvalue problem as follows

$$\underline{\underline{C}}^T \underline{\underline{K}} \underline{\underline{u}} \underline{\underline{C}} = \lambda \underline{\underline{C}}^T \underline{\underline{M}} \underline{\underline{C}} \underline{\underline{u}} \tag{15}$$

The T superscript indicates transpose. The transformed problem has the same eigenvalues as the original problem excluding the rigid body eigenvalue.

The transformation matrix for the sample problem is

$$\underline{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -2 \end{bmatrix} \quad (16)$$

The transposed mass matrix is

$$\begin{aligned} & \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} \left(\frac{1}{6} \right) \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} \left(\frac{1}{6} \right) \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ -2 & 3 \end{bmatrix} \\ &= \left(\frac{1}{6} \right) \begin{bmatrix} 4 & 4 \\ 4 & 8 \end{bmatrix} \end{aligned} \quad (17)$$

The transposed stiffness matrix is

$$\begin{aligned} & \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 4 \\ -1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 \\ 2 & 10 \end{bmatrix} \end{aligned} \quad (18)$$

The transposed eigenvalue problem is thus

$$\begin{bmatrix} 2 & 2 \\ 2 & 10 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \left(\frac{\hat{\lambda}}{6} \right) \begin{bmatrix} 4 & 4 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (19)$$

The resulting eigenvalues are

$$\begin{bmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \end{bmatrix} \quad (20)$$

The matrix deflation thus successfully removed the zero eigenvalue, which corresponded to the rigid-body mode.

The mass-normalized eigenvectors for the reduced system are

$$\bar{U}_c = \begin{bmatrix} 1.2247 & -1.2247 \\ 0 & 1.2247 \end{bmatrix} \quad (21)$$

The mass-normalized eigenvectors for the elastic modes of complete system are

$$\bar{U} = \underline{C} \bar{U}_c \quad (22)$$

$$\bar{U} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1.2247 & -1.2247 \\ 0 & 1.2247 \end{bmatrix} = \begin{bmatrix} 1.2247 & -1.2247 \\ 0 & 1.2247 \\ -1.2247 & -1.2247 \end{bmatrix} \quad (23)$$

Further Examples

The method is extended to beam bending in Appendix A, where two rigid-body modes must be removed from the eigenvalue problem.

References

1. L. Meirovitch, Analytical Methods in Vibrations, Macmillan, New York, 1967.
2. T. Irvine, Longitudinal Vibration of a Rod via the Finite Element Method, Rev A, Vibrationdata, 2003.
3. T. Irvine, Transverse Vibration of a Beam via the Finite Element Method, Vibrationdata, Rev D, 2004.

APPENDIX A

Matrix Deflation for Beam Bending

Consider the bending vibration for a free-free beam. Model the beam using the finite element method with one element. The generalized eigenvalue problem is

$$\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ h\theta_1 \\ y_2 \\ h\theta_2 \end{bmatrix} = \lambda \begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ h\theta_1 \\ y_2 \\ h\theta_2 \end{bmatrix} \quad (\text{A-1})$$

The mass and stiffness matrices in equation (A-1) are taken from Reference 3.

The eigenvalues are

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.7143 \\ 20 \end{bmatrix} \quad (\text{A-2})$$

Let

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \quad (\text{A-3})$$

Consider the translational rigid-body mode. The eigenvector is

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad (\text{A-4})$$

Derive the constraint equation for the translational rigid-body mode.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = 0 \tag{A-5}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} m_{11}u_1 + m_{12}u_2 + m_{13}u_3 + m_{14}u_4 \\ m_{21}u_1 + m_{22}u_2 + m_{23}u_3 + m_{24}u_4 \\ m_{31}u_1 + m_{32}u_2 + m_{33}u_3 + m_{34}u_4 \\ m_{41}u_1 + m_{42}u_2 + m_{43}u_3 + m_{44}u_4 \end{bmatrix} = 0 \tag{A-6}$$

$$\begin{aligned} & m_{11}u_1 + m_{12}u_2 + m_{13}u_3 + m_{14}u_4 \\ & + m_{31}u_1 + m_{32}u_2 + m_{33}u_3 + m_{34}u_4 = 0 \end{aligned} \tag{A-7}$$

The resulting constraint equation is thus

$$(m_{11} + m_{31})u_1 + (m_{12} + m_{32})u_2 + (m_{13} + m_{33})u_3 + (m_{14} + m_{34})u_4 = 0 \tag{A-8}$$

$$u_4 = - \left(\frac{m_{11} + m_{31}}{m_{14} + m_{34}} \right) u_1 - \left(\frac{m_{12} + m_{32}}{m_{14} + m_{34}} \right) u_2 - \left(\frac{m_{13} + m_{33}}{m_{14} + m_{34}} \right) u_3 \tag{A-9}$$

$$u_4 = 6u_1 + u_2 + 6u_3 \tag{A-10}$$

Form the constraint matrix

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}_e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -6 & -1 & -6 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & -6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad (\text{A-11})$$

The constraint matrix for translation is

$$\underline{C}_T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 1 & 6 \end{bmatrix} \quad (\text{A-12})$$

Transform the mass matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 1 & 6 \end{bmatrix} \\ = \begin{bmatrix} 144 & 15 & -12 \\ 15 & 2 & -3 \\ -12 & -3 & 36 \end{bmatrix} \quad (\text{A-12})$$

Transform the stiffness matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 1 & 6 \end{bmatrix} \\
= \begin{bmatrix} 288 & 48 & 132 \\ 48 & 12 & 24 \\ 132 & 24 & 84 \end{bmatrix} \tag{A-13}$$

The reduced generalized eigenvalue problem is

$$\begin{bmatrix} 288 & 48 & 132 \\ 48 & 12 & 24 \\ 132 & 24 & 84 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \lambda \begin{bmatrix} 144 & 15 & -12 \\ 15 & 2 & -3 \\ -12 & -3 & 36 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{A-14}$$

The eigenvalues are

$$\lambda = \begin{bmatrix} 0 \\ 1.714 \\ 20 \end{bmatrix} \tag{A-15}$$

Thus, the translation rigid-body mode has been successfully removed from the generalized eigenvalue problem.

Consider the rotational rigid-body mode. Determine the eigenvector by setting the eigenvalue to zero in equation (A-14). The eigenvector is

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \tag{A-16}$$

Derive the constraint equation for the rotational rigid-body mode using the matrices with the translation rigid-body mode already removed.

$$\begin{bmatrix} 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0 \quad (\text{A-17})$$

$$\begin{bmatrix} 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} m_{11}u_1 + m_{12}u_2 + m_{13}u_3 \\ m_{21}u_1 + m_{22}u_2 + m_{23}u_3 \\ m_{31}u_1 + m_{32}u_2 + m_{33}u_3 \end{bmatrix} = 0 \quad (\text{A-18})$$

$$\begin{aligned} & m_{11}u_1 + m_{12}u_2 + m_{13}u_3 \\ & -2m_{21}u_1 - 2m_{22}u_2 - 2m_{23}u_3 \\ & -m_{31}u_1 - m_{32}u_2 - m_{33}u_3 = 0 \end{aligned} \quad (\text{A-19})$$

$$\begin{aligned} & (m_{11} - 2m_{21} - m_{31})u_1 \\ & + (m_{12} - 2m_{22} - m_{32})u_2 \\ & + (m_{13} - 2m_{23} - m_{33})u_3 = 0 \end{aligned} \quad (\text{A-20})$$

$$u_3 = -\frac{(m_{11} - 2m_{21} - m_{31})}{(m_{13} - 2m_{23} - m_{33})}u_1 - \frac{(m_{12} - 2m_{22} - m_{32})}{(m_{13} - 2m_{23} - m_{33})}u_2 \quad (\text{A-21})$$

$$u_3 = 3u_1 + \frac{1}{3}u_2 \quad (\text{A-22})$$

The resulting constraint equation is

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 1/3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (\text{A-23})$$

The transformation matrix for rotation is

$$\underline{C_R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 1/3 \end{bmatrix} \quad (\text{A-24})$$

Transform the mass matrix.

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1/3 \end{bmatrix} \begin{bmatrix} 144 & 15 & -12 \\ 15 & 2 & -3 \\ -12 & -3 & 36 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 1/3 \end{bmatrix} \\ = \begin{bmatrix} 396 & 38 \\ 38 & 4 \end{bmatrix} \end{aligned} \quad (\text{A-24})$$

Transform the stiffness matrix.

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1/3 \end{bmatrix} \begin{bmatrix} 288 & 48 & 132 \\ 48 & 12 & 24 \\ 132 & 24 & 84 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 1/3 \end{bmatrix} \\ = \begin{bmatrix} 1.7760 & 0.2480 \\ 0.2480 & 0.0373 \end{bmatrix} \end{aligned} \quad (\text{A-25})$$

$$\begin{bmatrix} 1.7760 & 0.2480 \\ 0.2480 & 0.0373 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \lambda \begin{bmatrix} 396 & 38 \\ 38 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (\text{A-26})$$

The resulting eigenvalues are

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1.714 \\ 20 \end{bmatrix} \quad (\text{A-26})$$

The mass-normalized eigenvectors for the reduced system are

$$\bar{\mathbf{U}}_c = \begin{bmatrix} -0.1091 & -0.1291 \\ 0.6547 & 1.5492 \end{bmatrix} \quad (\text{A-27})$$

The mass-normalized eigenvectors for the elastic modes of complete system are

$$\bar{\mathbf{U}} = \underline{\mathbf{C}}_T \quad \underline{\mathbf{C}}_R \quad \bar{\mathbf{U}}_c \quad (\text{A-28})$$

$$\bar{\mathbf{U}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 1 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 1/3 \end{bmatrix} \begin{bmatrix} -0.1091 & -0.1291 \\ 0.6547 & 1.5492 \end{bmatrix} = \begin{bmatrix} -0.1091 & -0.1291 \\ 0.6547 & 1.5492 \\ -0.1091 & 0.1291 \\ -0.6547 & 1.5492 \end{bmatrix}$$

(A-29)