FREQUENCY RESPONSE FUNCTION FOR BEAM BENDING
WITH ENFORCED ACCELERATION VIA FINITE ELEMENT ANALYSIS

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Variables

A	Enforced acceleration amplitude
M	Mass matrix
K	Stiffness matrix
F	Applied forces
F_d	Forces at driven nodes
F_f	Forces at free nodes
I	Identity matrix
Π	Transformation matrix
u	Displacement vector
u_d	Displacements at driven nodes, time domain
u_f	Displacements at free nodes, time domain
$\hat{U}_d$	Displacements at driven nodes, frequency domain
$\hat{U}_f$	Displacements at free nodes, frequency domain

Derivation

The following method is adapted from Reference 1.

The equation of motion for a multi-degree-of-freedom system is

$$[M][\ddot{u}] + [K][u] = F \tag{1}$$

The displacement vector is

$$[u] = \begin{bmatrix} u_d \\ u_f \end{bmatrix} \tag{2}$$

Partition the matrices and vectors as follows

$$\begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix} \begin{bmatrix} \ddot{u}_d \\ \ddot{u}_f \end{bmatrix} + \begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix} \begin{bmatrix} u_d \\ u_f \end{bmatrix} = \begin{bmatrix} F_d \\ F_f \end{bmatrix} \tag{3}$$

Create a transformation matrix such that

$$\begin{bmatrix} u_d \\ u_f \end{bmatrix} = \Pi \begin{bmatrix} u_d \\ u_w \end{bmatrix} \tag{4}$$

$$\Pi = \begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix} \Pi \begin{bmatrix} \ddot{u}_d \\ \ddot{u}_w \end{bmatrix} + \begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix} \Pi \begin{bmatrix} u_d \\ u_w \end{bmatrix} = \begin{bmatrix} F_d \\ F_f \end{bmatrix} \tag{6}$$

Premultiply by Π^T,

$$\Pi^T \begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix} \Pi \begin{bmatrix} \ddot{u}_d \\ \ddot{u}_w \end{bmatrix} + \Pi^T \begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix} \Pi \begin{bmatrix} u_d \\ u_w \end{bmatrix} = \Pi^T \begin{bmatrix} F_d \\ F_f \end{bmatrix} \tag{7}$$

Transform the equation of motion to uncouple the stiffness matrix so that the resulting stiffness matrix is

$$\begin{bmatrix} \hat{K}_{dd} & 0 \\ 0 & \hat{K}_{ww} \end{bmatrix} \tag{8}$$

$$\Pi^T K \Pi = \begin{bmatrix} I & T_1^T \\ 0 & T_2^T \end{bmatrix} \begin{bmatrix} K_{dd} & K_{df} \\ K_{fd} & K_{ff} \end{bmatrix} \begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix} \tag{9}$$

$$\Pi^T K \Pi = \begin{bmatrix} I & T_1^T \\ 0 & T_2^T \end{bmatrix} \begin{bmatrix} K_{dd} + K_{df} T_1 & K_{df} T_2 \\ K_{fd} + K_{ff} T_1 & K_{ff} T_2 \end{bmatrix} \tag{10}$$

$$\Pi^T K \Pi = \begin{bmatrix} K_{dd} + K_{df} T_1 + T_1^T \left(K_{fd} + K_{ff} T\right) & K_{df} T_2 + T_1^T K_{ff} T_2 \\ T_2^T \left(K_{fd} + K_{ff} T_1\right) & T_2^T \left(K_{ff} T_2\right) \end{bmatrix} \tag{11}$$

$$\Pi^T K \Pi = \begin{bmatrix} K_{dd} + K_{df} T_1 + T_1^T \left(K_{fd} + K_{ff} T_1\right) & \left(K_{df} + T_1^T K_{ff}\right) T_2 \\ T_2^T \left(K_{fd} + K_{ff} T_1\right) & T_2^T \left(K_{ff} T_2\right) \end{bmatrix} \tag{12}$$

$$\Pi^T K \Pi = \begin{bmatrix} K_{dd} + T_1^T K_{fd} + \left(K_{df} + T_1^T K_{ff}\right) T_1 & \left(K_{df} + T_1^T K_{ff}\right) T_2 \\ T_2^T \left(K_{fd} + K_{ff} T_1\right) & T_2^T \left(K_{ff} T_2\right) \end{bmatrix} \tag{13}$$

Let

$$T_2 = I \tag{14}$$

$$\Pi^T K \Pi = \begin{bmatrix} K_{dd} + T_1{}^T K_{fd} + \left(K_{df} + T_1{}^T K_{ff}\right) T_1 & \left(K_{df} + T_1{}^T K_{ff}\right) \\ \left(K_{fd} + K_{ff} T_1\right) & K_{ff} \end{bmatrix} = \begin{bmatrix} \hat{K}_{dd} & 0 \\ 0 & \hat{K}_{ww} \end{bmatrix} \tag{15}$$

$$K_{df} + T_1{}^T K_{ff} = 0 \tag{16}$$

$$T_1{}^T = -K_{df} K_{ff}{}^{-1} \tag{17}$$

$$T_1 = -K_{ff}{}^{-1} K_{fd} \tag{18}$$

$$\Pi = \begin{bmatrix} I_{dd} & 0 \\ T_1 & I_{ff} \end{bmatrix} \tag{19}$$

$$\hat{K}_{dd} = K_{dd} + T_1{}^T K_{fd} + \left(K_{df} + T_1{}^T K_{ff}\right) T_1 \tag{20}$$

$$\hat{K}_{ww} = K_{ff} \tag{21}$$

$$\Pi^T M \Pi = \begin{bmatrix} I_{dd} & T_1{}^T \\ 0 & I_{ff} \end{bmatrix} \begin{bmatrix} M_{dd} & M_{df} \\ M_{fd} & M_{ff} \end{bmatrix} \begin{bmatrix} I_{dd} & 0 \\ T_1 & I_{ff} \end{bmatrix} \tag{22}$$

By similarity, the transformed mass matrix is

$$\begin{bmatrix} \hat{m}_{dd} & \hat{m}_{dw} \\ \hat{m}_{wd} & \hat{m}_{ww} \end{bmatrix} = \begin{bmatrix} M_{dd} + T_1{}^T M_{fd} + \left(M_{df} + T_1{}^T M_{ff} \right) T_1 & \left(M_{df} + T_1{}^T M_{ff} \right) \\ \left(M_{fd} + M_{ff} T_1 \right) & M_{ff} \end{bmatrix} \tag{23}$$

$$\begin{bmatrix} \widehat{F}_d \\ \hat{F}_w \end{bmatrix} = \begin{bmatrix} I_{dd} & T_1 \\ 0 & I_{ff} \end{bmatrix} \begin{bmatrix} F_d \\ F_f \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} \widehat{F}_d \\ \hat{F}_w \end{bmatrix} = \begin{bmatrix} I_{dd} F_d + T_1 F_f \\ I_{ff} F_f \end{bmatrix} \tag{25}$$

$$\begin{bmatrix} \widehat{F}_d \\ \hat{F}_w \end{bmatrix} = \begin{bmatrix} F_d + T_1 F_f \\ F_f \end{bmatrix} \tag{26}$$

$$\begin{bmatrix} \hat{m}_{dd} & \hat{m}_{dw} \\ \hat{m}_{wd} & \hat{m}_{ww} \end{bmatrix} \begin{bmatrix} \ddot{u}_d \\ \ddot{u}_w \end{bmatrix} + \begin{bmatrix} \hat{K}_{dd} & 0 \\ 0 & \hat{K}_{ww} \end{bmatrix} \begin{bmatrix} u_d \\ u_w \end{bmatrix} = \begin{bmatrix} \widehat{F}_d \\ \hat{F}_w \end{bmatrix} \tag{27}$$

$$\hat{m}_{wd} \ddot{u}_d + \hat{m}_{ww} \ddot{u}_w + \hat{K}_{ww} u_w = \hat{F}_w \tag{28}$$

The equation of motion is thus

$$\hat{m}_{ww} \ddot{u}_w + \hat{K}_{ww} u_w = \hat{F}_w - \hat{m}_{wd} \ddot{u}_d \tag{29}$$

Assume that the external forces are zero.

$$\hat{m}_{ww} \ddot{u}_w + \hat{K}_{ww} u_w = -\hat{m}_{wd} \ddot{u}_d \tag{30}$$

Consider the homogeneous form of equation (30).

$$\hat{m}_{ww}\ddot{u}_w + \hat{K}_{ww}u_w = \bar{0} \tag{31}$$

Seek a solution of the form

$$\bar{u}_w = \bar{q}\exp(j\omega t) \tag{32}$$

The q vector is the generalized coordinate vector.

Note that

$$\dot{\bar{u}} = j\omega\,\bar{q}\exp(j\omega t) \tag{33}$$

$$\ddot{\bar{u}} = -\omega^2\,\bar{q}\exp(j\omega t) \tag{34}$$

By substitution,

$$-\omega^2\hat{m}_{ww}\;\bar{q}\exp(j\omega t) + \hat{K}_{ww}\;\bar{q}\exp(j\omega t) = \bar{0} \tag{35}$$

$$\left\{-\omega^2\hat{m}_{ww}\;\bar{q} + \hat{K}_{ww}\;\bar{q}\right\}\exp(j\omega t) = \bar{0} \tag{36}$$

$$\left\{-\omega_n{}^2\hat{m}_{ww}\;\bar{q} + \hat{K}_{ww}\;\bar{q}\right\}\exp(j\omega_n t) = \bar{0} \tag{37}$$

$$\left\{-\omega^2\hat{m}_{ww} + \hat{K}_{ww}\right\}\bar{q} = \bar{0} \tag{38}$$

$$\left\{\hat{K}_{ww} - \omega^2\hat{m}_{ww}\right\}\bar{q} = \bar{0} \tag{39}$$

Equation (39) is an example of a generalized eigenvalue problem. The eigenvalues can be found by setting the determinant equal to zero.

$$\det\left\{\hat{K}_{ww} - \omega^2\hat{m}_{ww}\right\} = 0 \tag{40}$$

The eigenvectors are found via the following equations.

$$\left\{ \hat{K}_{ww} - \omega_i{}^2 \, \hat{m}_{ww} \right\} \bar{q}_i = \bar{0} \tag{41}$$

An eigenvector matrix Q can be formed. The eigenvectors are inserted in column format.

$$Q = \left[\, \bar{q}_1 \;\; | \;\; \bar{q}_2 \;\; | \;\; \ldots \;\; | \;\; \bar{q}_n \right] \tag{42}$$

where n is the number of degrees-of-freedom

The eigenvectors represent orthogonal mode shapes. Assume that the eigenvectors are mass-normalized such that

$$Q^T M Q = I \tag{43}$$

and

$$Q^T K Q = \Omega \tag{44}$$

where

superscript T represents transpose

I is the identity matrix

Ω is a diagonal matrix of eigenvalues

Now define a modal coordinate $\eta(t)$ such that

$$\bar{u} = Q \, \bar{\eta} \tag{45}$$

Let q_{ij} represent the elements of Q.

The displacement, velocity and acceleration terms are

$$u_i = \sum_{j=1}^{n} q_{ij} \, \eta_j \tag{46}$$

$$\dot{u}_i = \sum_{j=1}^{n} q_{ij}\,\dot{\eta}_j \tag{47}$$

$$\ddot{u}_i = \sum_{j=1}^{n} q_{ij}\,\ddot{\eta}_j \tag{48}$$

By substitution

$$\hat{m}_{ww}\ Q\ \ddot{\bar{\eta}} + \hat{K}_{ww}\ Q\ \bar{\eta} = -\hat{m}_{wd}\,\ddot{u}_d \tag{49}$$

Premultiply by the transpose of the normalized eigenvector matrix.

$$Q^T\,\hat{m}_{ww}\ Q\ \ddot{\bar{\eta}}\ +\ Q^T\hat{K}_{ww}\ Q\ \bar{\eta} = -\hat{Q}^T\hat{m}_{wd}\,\ddot{u}_d \tag{50}$$

The orthogonality relationships yield

$$I\ \ddot{\bar{\eta}}\ +\ \Omega\ \bar{\eta} = -\hat{Q}^T\hat{m}_{wd}\,\ddot{u}_d \tag{51}$$

Note that the two equations are decoupled in terms of the modal coordinate.

Now assume modal damping by adding an uncoupled damping matrix.

$$I\ \ddot{\bar{\eta}}\ + D\dot{\bar{\eta}} + \Omega\ \bar{\eta} = -Q^T\hat{m}_{wd}\,\ddot{u}_d \tag{52}$$

$$D_{ij} = \begin{cases} 2\xi_i\omega_i{}^2, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases} \tag{53}$$

Now assume a harmonic base input. Assume that it is uniform if it is applied at multiple locations.

$$\ddot{y} = A \exp(j\omega t) \tag{54}$$

Assume a common harmonic modal displacement.

$$\eta_i = \psi_i \exp(j\omega t) \tag{55}$$

$$\dot{\eta}_i = j\,\omega_i\,\psi_i \exp(j\omega t) \tag{56}$$

$$\ddot{\eta}_i = -\omega_i^2\,\psi_i \exp(j\omega t) \tag{57}$$

Let C be a vectors of ones. The number of rows in C is equal to the number of drive points.

By substitution,

$$\left\{-\omega^2 + j\,2\,\xi_i\,\omega_i\,\omega + \omega_i^2\right\}\psi_i \exp(j\omega t) = -\left\{\left\lfloor Q^T \hat{m}_{wd} \right\rfloor_{row\,i} C\right\} A \exp(j\omega t) \tag{58}$$

$$\left\{\left[\omega_i^2 - \omega^2\right] + j\,2\,\xi_i\,\omega_i\,\omega\right\}\psi_i \exp(j\omega t) = -\left\{\left\lfloor Q^T \hat{m}_{wd} \right\rfloor_{row\,i} C\right\} A \exp(j\omega t) \tag{59}$$

The modal displacement is

$$\eta_i = \psi_i \exp(j\omega t) = \frac{-\left\{\left\lfloor Q^T \hat{m}_{wd} \right\rfloor_{row\,i} C\right\}}{\left\{\left[\omega_i^2 - \omega^2\right] + j\,2\,\xi_i\,\omega_i\,\omega\right\}} A \exp(j\omega t) \tag{60}$$

The modal velocity is

$$\dot{\eta}_i = \frac{-j\omega \left\{ \left[Q^T \hat{m}_{wd} \right]_{row\,i} C \right\}}{\left\{ \left[\omega_i^2 - \omega^2 \right] + j\, 2 \xi_i\, \omega_i\, \omega \right\}} A \exp(j\omega t) \tag{61}$$

The modal acceleration is

$$\ddot{\eta}_i = \frac{\omega^2 \left\{ \left[Q^T \hat{m}_{wd} \right]_{row\,i} C \right\}}{\left\{ \left[\omega_i^2 - \omega^2 \right] + j\, 2 \xi_i\, \omega_i\, \omega \right\}} A \exp(j\omega t) \tag{62}$$

Recall

$$\ddot{u}_i = \sum_{p=1}^{n} q_{ip}\, \ddot{\eta}_p \tag{63}$$

$$\ddot{u}_i = \sum_{p=1}^{n} \left\{ q_{ip} \frac{\omega^2 \left\{ \left[Q^T \hat{m}_{wd} \right]_{row\,i} C \right\}}{\left\{ \left[\omega_p^2 - \omega^2 \right] + j\, 2 \xi_p\, \omega_p\, \omega \right\}} A \exp(j\omega t) \right\} \tag{64}$$

$$\ddot{u}_i = A \exp(j\omega t) \sum_{p=1}^{n} \left\{ q_{ip} \frac{\omega^2 \left\{ \left[Q^T \hat{m}_{wd} \right]_{row\,i} C \right\}}{\left\{ \left[\omega_p^2 - \omega^2 \right] + j\, 2 \xi_p\, \omega_p\, \omega \right\}} \right\} \tag{65}$$

The Fourier transform equation is

$$\hat{U}_i(f) = \int_{-\infty}^{\infty} \ddot{u}_i(t) \exp[-j\omega t]\, dt \tag{66}$$

$$\ddot{u}_i = A \exp(j\omega t) \sum_{p=1}^{n} \left\{ q_{ip} \frac{\omega^2 \left\{ \left[Q^T \hat{m}_{wd} \right]_{row\,i} C \right\}}{\left\{ \left[\omega_p{}^2 - \omega^2 \right] + j\, 2 \xi_p \omega_p \omega \right\}} \right\} \tag{67}$$

$$\hat{U}_i(f)/A = \sum_{p=1}^{n} \left\{ q_{ip} \frac{\omega^2 \left\{ \left[Q^T \hat{m}_{wd} \right]_{row\,i} C \right\}}{\left\{ \left[\omega_p{}^2 - \omega^2 \right] + j\, 2 \xi_p \omega_p \omega \right\}} \right\} \tag{68}$$

Recall

$$\begin{bmatrix} u_d \\ u_f \end{bmatrix} = \Pi \begin{bmatrix} u_d \\ u_w \end{bmatrix} \tag{69}$$

The equivalent format for the frequency domain is

$$\begin{bmatrix} \hat{U}_d \\ \hat{U}_f \end{bmatrix} = \Pi \begin{bmatrix} \hat{U}_d \\ \hat{U}_w \end{bmatrix} \tag{70}$$

The final step is to rearrange the degrees-of-freedom in the proper order.

References

1. T. Irvine, Modal Transient Analysis of a Multi-degree-of-freedom System with Enforced Motion, Revision E, Vibrationdata, 2012.

2. T. Irvine, Transverse Vibration of a Beam via the Finite Element Method, Revision F, Vibrationdata, 2010.

APPENDIX A

Example

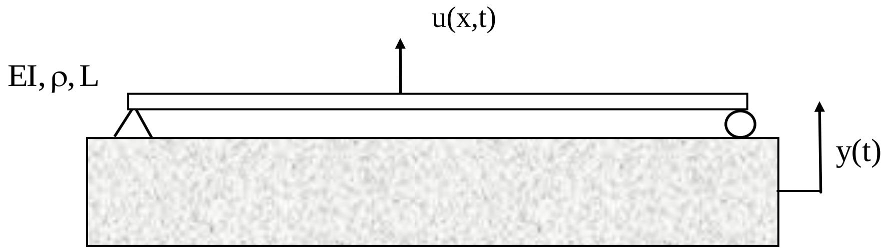

Figure A-1.

Consider a beam subjected to uniform base excitation at each end.

The beam has the following properties:

Cross-Section	Circular
Boundary Conditions	Simply-Supported at Each End
Material	Aluminum

Diameter	D	=	0.5 inch
Cross-Section Area	A	=	0.1963 in^2
Length	L	=	24 inch
Area Moment of Inertia	I	=	0.003068 in^4
Elastic Modulus	E	=	1.0e+07 lbf/in^2
Stiffness	EI	=	30680 lbf in^2
Mass per Volume	ρ_V	=	0.1 lbm / in^3 (0.000259 lbf sec^2/in^4)
Mass per Length	ρ	=	0.01963 lbm/in (5.08e-05 lbf sec^2/in^2)
Mass	ρL	=	0.471 lbm (1.22E-03 lbf sec^2/in)
Viscous Damping Ratio	ξ	=	0.05

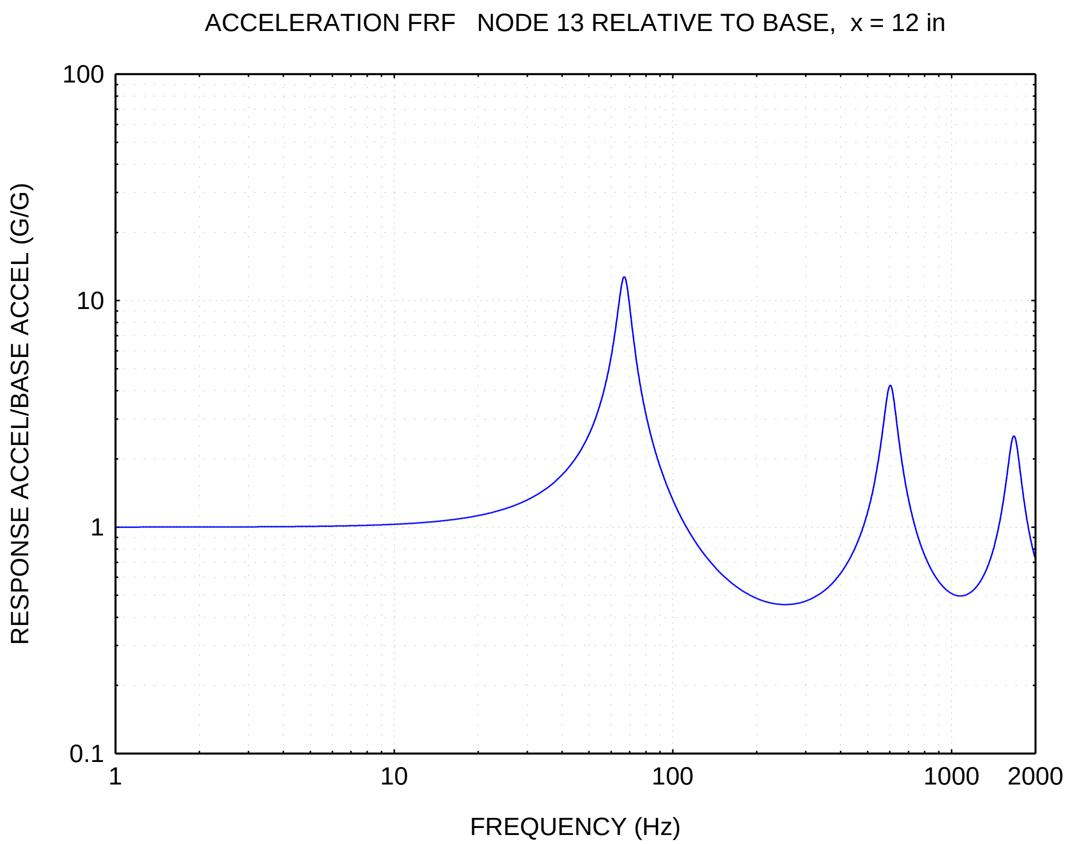

Figure A-2.

Model the beam with 24 elements using the method in Reference 2. Calculate the frequency response function at the center.