THE PERIOD OF A PENDULUM UNDERGOING LARGE DEFLECTIONS

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Introduction

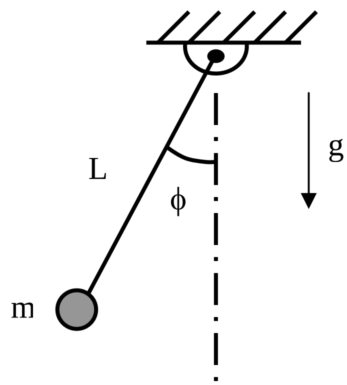

Figure 1.

Consider a simple pendulum consisting of a point mass connected by a rod to a pivot point, where the rod is mass-less.

The derivation of the equation of motion is given in Reference 1.

The angular displacement ϕ of the pendulum for free vibration is governed by the nonlinear differential equation.

$$mL^2\ddot{\phi} + mgL\sin\phi = 0 \tag{1}$$

where

m	=	Mass
L	=	Length
g	=	Gravity acceleration

Divide through by mL^2.

$$\ddot{\phi} + \frac{g}{L} \sin\phi = 0 \qquad (2)$$

and

$$\omega = \sqrt{\frac{g}{L}} \qquad (3)$$

$$\ddot{\phi} + \omega^2 \sin\phi = 0 \qquad (4)$$

The following pair of equations for the period τ of a fully cycle are taken from Reference 1, equation (2.7).

$$\ddot{\phi} + \omega^2 f(\phi) = 0 \qquad (5)$$

$$\tau = \frac{4}{\omega} \int_0^{\phi_m} \frac{d\phi}{\sqrt{2\int_{\phi}^{\phi_m} f(\phi')d\phi'}} \qquad (6)$$

where

ϕ	is the displacement
ϕ_m	is the maximum displacement

The maximum displacement must be found numerically solving equation (4).

$$\tau = \frac{4}{\omega} \int_0^{\phi_m} \frac{d\phi}{\sqrt{2\int_{\phi}^{\phi_m} \sin(\phi')d\phi'}} \qquad (7)$$

$$\tau = \frac{4}{\omega} \int_0^{\phi_m} \frac{d\phi}{\sqrt{-2\cos(\phi')\Big|_{\phi}^{\phi_m}}} \qquad (8)$$

$$\tau = \frac{4}{\omega\sqrt{2}} \int_0^{\phi_m} \frac{d\phi}{\sqrt{-\cos(\phi')\Big|_{\phi}^{\phi_m}}} \qquad (9)$$

$$\tau = \frac{4}{\omega\sqrt{2}}\int_0^{\phi_m} \frac{d\phi}{\sqrt{\cos(\phi) - \cos(\phi_m)}} \tag{10}$$

Trigonometric identity

$$\cos(\phi) - \cos(\phi_m) = 2\left[\sin^2\left(\frac{\phi_m}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)\right] \tag{11}$$

Substitute equation (11) into equation (10).

$$\tau = \frac{4}{\omega\sqrt{2}}\int_0^{\phi_m} \frac{d\phi}{\sqrt{2\left[\sin^2\left(\frac{\phi_m}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)\right]}} \tag{12}$$

$$\tau = \frac{2}{\omega}\int_0^{\phi_m} \frac{d\phi}{\sqrt{\sin^2\left(\frac{\phi_m}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)}} \tag{13}$$

$$\tau = \frac{2}{\omega}\int_0^{\phi_m} \left\{\sqrt{\sin^2\left(\frac{\phi_m}{2}\right)}\sqrt{1 - \frac{\sin^2\left(\frac{\phi}{2}\right)}{\sin^2\left(\frac{\phi_m}{2}\right)}}\right\}^{-1} d\phi \tag{14}$$

$$\tau = \frac{2}{\omega \sin\left(\frac{\phi_m}{2}\right)}\int_0^{\phi_m} \left\{\sqrt{1 - \frac{\sin^2\left(\frac{\phi}{2}\right)}{\sin^2\left(\frac{\phi_m}{2}\right)}}\right\}^{-1} d\phi \tag{15}$$

Let

$$z = \frac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi m}{2}\right)} \tag{16}$$

$$z\sin\left(\frac{\phi m}{2}\right) = \sin\left(\frac{\phi}{2}\right) \tag{17}$$

$$z^2\sin^2\left(\frac{\phi m}{2}\right) = \sin^2\left(\frac{\phi}{2}\right) \tag{18}$$

$$1 - z^2\sin^2\left(\frac{\phi m}{2}\right) = 1 - \sin^2\left(\frac{\phi}{2}\right) \tag{19}$$

$$\sqrt{1 - z^2\sin^2\left(\frac{\phi m}{2}\right)} = \sqrt{1 - \sin^2\left(\frac{\phi}{2}\right)} \tag{20}$$

$$\sqrt{1 - z^2\sin^2\left(\frac{\phi m}{2}\right)} = \cos\left(\frac{\phi}{2}\right) \tag{21}$$

Now take the derivative of equation (16).

$$dz = \frac{\cos\left(\frac{\phi}{2}\right)}{2\sin\left(\frac{\phi m}{2}\right)}d\phi \tag{22}$$

$$dz = \frac{\sqrt{1 - z^2\sin^2\left(\frac{\phi m}{2}\right)}}{2\sin\left(\frac{\phi m}{2}\right)}d\phi \tag{23}$$

$$\frac{2\sin\left(\frac{\phi m}{2}\right)}{\sqrt{1-z^2\sin^2\left(\frac{\phi m}{2}\right)}}\,dz = d\phi \qquad (24)$$

Recall

$$z = \frac{\sin\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi m}{2}\right)} \qquad (25)$$

Substitute equations (24) and (25) into (15).

$$\tau = \frac{2}{\omega\sin\left(\frac{\phi m}{2}\right)}\int_0^1 \frac{2\sin\left(\frac{\phi m}{2}\right)}{\sqrt{1-z^2\sin^2\left(\frac{\phi m}{2}\right)}\,\sqrt{1-z^2}}\,dz \qquad (26)$$

$$\tau = \frac{4}{\omega}\int_0^1 \frac{dz}{\sqrt{1-z^2\sin^2\left(\frac{\phi m}{2}\right)}\,\sqrt{1-z^2}} \qquad (27)$$

Let

$$k = \sin\left(\frac{\phi m}{2}\right) \qquad (28)$$

$$\tau = \frac{4}{\omega}\int_0^1 \frac{dz}{\sqrt{1-k^2 z^2}\,\sqrt{1-z^2}} \qquad (29)$$

$$\tau = \frac{4}{\omega}\int_0^1 \frac{dz}{\sqrt{\left[1-k^2 z^2\right]\left[1-z^2\right]}} \qquad (30)$$

Equation (30) is known as the elliptic integral of the first kind. The names Jacobi and Legendre are often associated with this integral.

The detailed solution leads to an elliptic integral. That solution can be approximated by a series, as taken from Reference 3.

The complete elliptic integral of the first kind *K* is defined as

$$K(k) = \int_0^1 \frac{1}{\sqrt{\left(1-z^2\right)\left(1-k^2 z^2\right)}}\, dz \tag{31}$$

The solution is

$$K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[k^{2n} \frac{(2n)!(2n)!}{16^n\, n!\, n!\, n!\, n!} \right] \tag{32}$$

$$K(k) = \frac{\pi}{2} \left\{ 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \bullet 3}{2 \bullet 4}\right)^2 k^4 + \ldots + \left[\frac{(2n-1)!!}{2^n\, n!}\right]^2 k^{2n} + \ldots \right\} \tag{33}$$

The period of the pendulum is thus

$$\tau = \frac{4}{\omega}\left[\frac{\pi}{2}\right]\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \frac{25}{256}k^6 + \ldots\right] \tag{34}$$

Again,

$$k = \sin\left(\frac{\phi_m}{2}\right) \tag{35}$$

References

1. T. Irvine, Pendulum Oscillation, Revision C, Vibrationdata, 1999.

2. Weaver, Timoshenko, and Young; Vibration Problems in Engineering, Wiley-Interscience, New York, 1990.

3. http://en.wikipedia.org/wiki/Elliptic_integral