A PINK NOISE FILTER  Revision A

By Tom Irvine
Email:  tomirvine@aol.com

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## Colors of Noise

Signals can be represented in the frequency domain by a power spectrum, where power is proportional to amplitude squared.

The spectrum coordinates can be represented in terms of a constant bandwidth, such as a 5 Hz increment.  As an alternative, the spectrum can be represented in terms of a proportional bandwidth, such as octave bands.

Two random signals of particular interest are white noise and pink noise.

White noise is a random signal which has a constant power spectrum for a constant frequency bandwidth.  It is thus analogous to white light, which is composed of a continuous spectrum of colors.

Pink noise is a random signal which has a constant power spectrum for each octave band.  This noise is called pink because the low frequency or “red” end of the spectrum is emphasized.  Pink noise is used in acoustics to measure the frequency response of an audio system in a particular room.  It can thus be used to calibrate a graphic equalizer.

A pink noise power spectrum plotted with respect to a constant bandwidth would have a 3-dB/octave roll-off.  This is equivalent to a slope of –1 on a power spectrum plot in log-log format.

A pink noise power spectrum is thus proportional to 1/f, where f is the frequency.  Again, power is proportional to the amplitude squared.  The pink noise amplitude must thus be proportional to 1/sqrt(f).  Note that each of these cases is equivalent to a 3 dB/octave roll-off.

A pink noise signal can thus be obtained by lowpass filtering white noise, where the filter has a 3 dB/octave roll-off.

Thee purpose of this report is to present methods for implementing this filter.  The frequency domain of concern is the audio domain, which extends to 20 kHz.

## Convolution Integral

A convolution integral method is given in Appendix A.  The convolution integral can be converted to a series for digital data input.  This method is rather inefficient, however, for digital computation.

Unfortunately, the convolution integral for a pink noise filter does not have a corresponding analog circuit. An approximate approach is required.

<u>Analog Filter Circuit</u>

A schematic of a lowpass filter with an approximate 3 dB/octave roll-off is shown in Figure 1 and by the values in Table 1. The schematic is taken from Reference 1. The transfer function involves cascading several stages so that the zeros of one stage partially cancel the poles of the next.

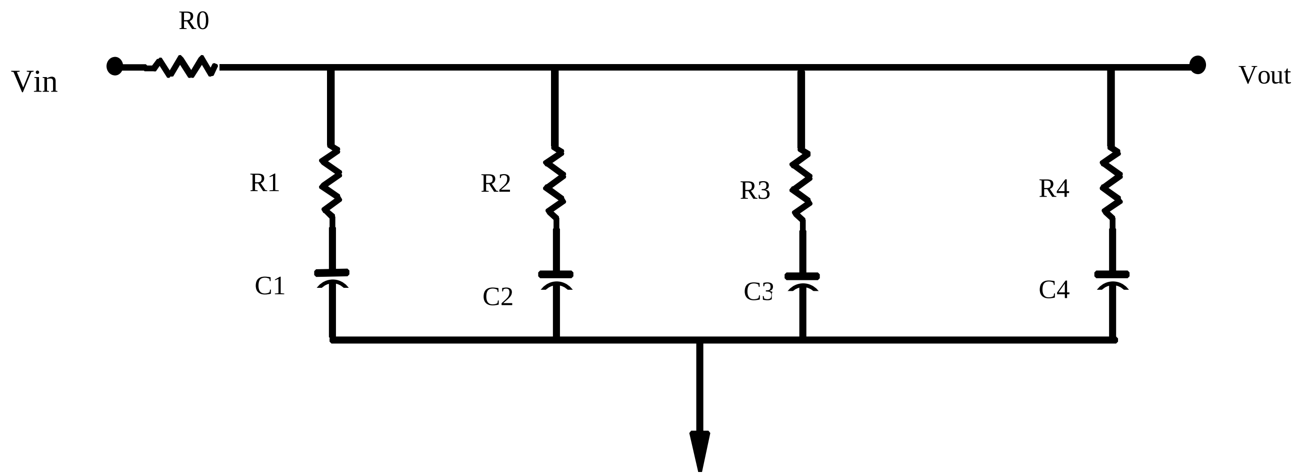


Figure 1. Analog Filter Circuit with 3 dB/octave Roll-off

| Table 1. Analog Filter Circuit | |
|---|---|
| Component | Value |
| R0 | 5.6 kΩ |
| R1 | 2.7 kΩ |
| R2 | 820 Ω |
| R3 | 270 Ω |
| R4 | 0 |
| C1 | 1.0 µF |
| C2 | 0.316 µF |
| C3 | 0.1 µF |
| C4 | 0.05 µF |

The circuit in Figure 1 is offered as an empirical approach.

The resulting transfer magnitude is given in equation 1. The coefficients are given in Table 2. A derivation of the transfer function is given in Appendix B.

$$H(s) = \frac{A_n s^4 + B_n s^3 + C_n s^2 + D_n s + E_n}{A_d s^4 + B_d s^3 + C_d s^2 + D_d s + E_d} \qquad (1)$$

Table 2.
Transfer Function Coefficients

| Coefficient | Value |
|---|---|
| An | 0 |
| Bn | 1.19556e+016 |
| Cn | 4.933672e+020 |
| Dn | 1.889949e+024 |
| En | 6.329114e+026 |
| Ad | 3.347568e+012 |
| Bd | 5.045112e+017 |
| Cd | 6.139691e+021 |
| Dd | 7.085899e+024 |
| Ed | 6.329114e+026 |

The transfer magnitude is plotted in Figure 1. The audio frequency domain is represented, up to 20 kHz. The transfer magnitude has an approximate roll-off of 3 dB/octave.

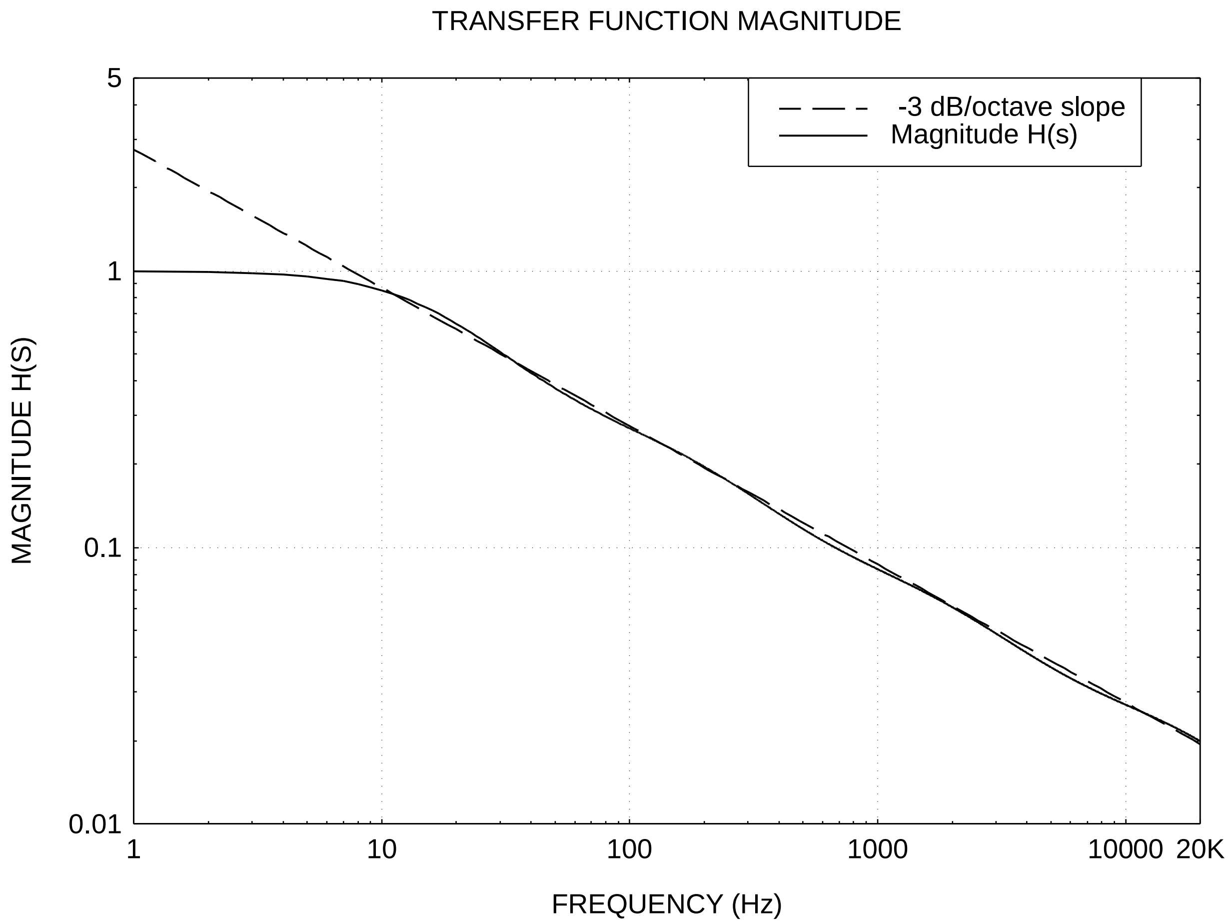


Figure 1. Analog Circuit Transfer Function

<u>Conclusion</u>

The purpose of this report was to determine a filter with a 3 dB/octave roll-off. This filter would allow white noise to be transformed to pink noise.

The convolution integral approach was briefly considered. A passive analog circuit was also considered.

A particular goal of the analysis was to derive an equivalent digital recursive filtering relationship to represent the transfer function of the analog circuit. The recursive relationship is desired for computational efficiency.

A relationship was derived, as shown in Appendix B. The accuracy of the algorithm is sensitive to the sampling rate, however. A minimum sampling rate of 1 million samples per second is required to obtain a reasonably accurate result. This rate was determined by a trial-and-error process. The intermediate results are omitted for brevity.

Further effort is required to obtain a more robust recursive relationship.

An additional method would be to manipulate the Fourier transform of the input signal. This method was not considered in this report, but may be considered in a future revision.

References


1. F. Everest, The Master Handbook of Acoustics, Tab, Blue Ridge Summit, PA, 1981.
2. F. Oberhettinger and L. Badii, Table of Laplace Transforms, Springer-Verlag, N.Y., 1972.
3. Stearns and David, Signal Processing Algorithms in Fortran and C, Prentice Hall, Englewood Cliffs, New Jersey, 1993.
4. Dorf, Modern Control Systems, Addison-Wesley, Reading, Massachusetts, 1980.
5. Ken Steiglitz, A Digital Signal Processing Primer, Addison-Wesley, Reading, Massachusetts, 1996

APPENDIX A

Convolution Approach

The Laplace transform

$$F(s) = \frac{1}{\sqrt{s+a}} \tag{A-1}$$

has an inverse transform of

$$f(t) = \frac{1}{\sqrt{\pi t}} \exp[-at] \tag{A-2}$$

The inverse is taken from Reference 2.

Equation (A-2) is thus the impulse response function for a pink noise filter.

Assume a transfer function of

$$\hat{H}(s) = \frac{3}{\sqrt{s+8\pi}} \qquad \text{(A-1)}$$

This transfer function is lowpass filter with a roll-off of approximately 3 dB/octave. The constant in the denominator is used to avoid a discontinuity at s equal to zero.

The impulse response function is found using equation (A-2).

$$\hat{h}(t) = \frac{3}{\sqrt{\pi t}} \exp[-8\pi t] \qquad \text{(A-2)}$$

Let x(t) be a white noise time history. The pink noise time history y(t) can be calculated via a convolution integral.

$$y(t) = \int_0^t \hat{h}(t-\tau)x(\tau)d\tau \tag{A-3}$$

The equivalent series form for digital data with a constant sample rate is

$$y_n = \sum_{i=0}^{n} \left[ \hat{h}\left(t_n - \tau_i\right) x_i \, \Delta\tau \right] \tag{A-4}$$

By substitution,

$$y_n = \sum_{i=0}^{n-1} \left[ \frac{3}{\sqrt{\pi\left(t_n - \tau_i\right)}} \exp\left[-8\pi\left(t_n - \tau_i\right)\right] x_i \, \Delta\tau \right] \tag{A-5}$$

The upper limit is changed to n-1 to avoid a discontinuity.

Equation (A-5) is very inefficient to calculate, however. Unfortunately, it does not appear to be adaptable to a recursive form. Consider a time series with 10,000 points. Equation (A-5) shows there would be 50 million series addition steps alone. The number of operations increases greatly when the evaluation of the square root function, the evaluation of the exponential function, and the associated multiplication steps are included.

A computer program for solving equation (A-5) is given below. It is written in C. The input file must be called a.in. It must have two columns: time and amplitude. The output file is called a.out. The output will be pink noise if the input is white noise. A white noise program is given in Appendix C.

```
#include <math.h>
#include <string.h>
#include <stdio.h>
#include <stdlib.h>

#define MAX 40000

long i,j,last;

const double pi=atan2(0.,-1.);
const double eightpi=8.*pi;

double t[MAX],a[MAX];

double dt,tt,x;

char filename[4][10];
FILE *pFile[4];

void main()
{
```

```
    strcpy(filename[0],"a.in");
    strcpy(filename[1],"a.out");

    pFile[0]=fopen(filename[0],"rb");
    pFile[1]=fopen(filename[1],"w");

    i=0;

    while( fscanf(pFile[0],"%lf %lf",&t[i],&a[i])>0 )
    {
        i++;
    }
    last=i;

    if( last > 2)
    {
        dt=(t[last-1]-t[0])/(double(last-1));

        for(i=0; i < last; i++)
        {
            x=0.;

            for(j=0; j < i; j++)
            {
                tt=(i-j)*dt;
                x+=(a[j]*exp(-eightpi*tt))/sqrt(pi*tt);
            }

            x*=3.*dt;

            fprintf(pFile[1],"%14.7e %14.7e \n",t[i],x);
        }
    }
}
```

APPENDIX B

Derivation of Recursive Relationship

The circuit in Figure 1 can be represented in terms of impedance elements as shown in Figure B-1 and in Table B-1. The impedance terms are a function of the Laplace variable s.

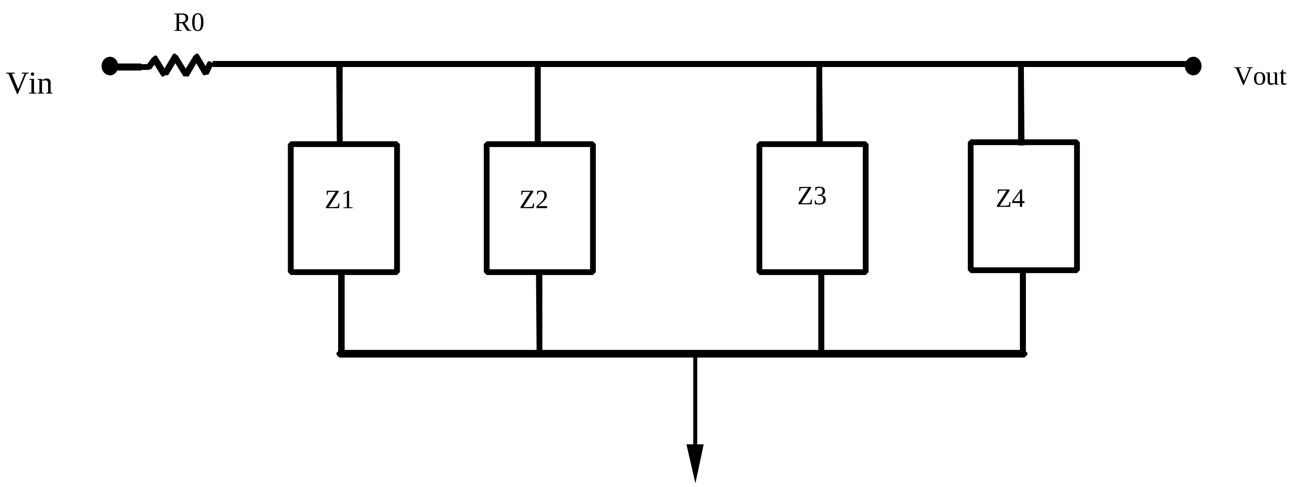


Figure B-1. Filter Circuit with Impedance Elements

Table B-1.
Impedance Values

| Element | Value | Alternate Form |
|---|---|---|
| $Z_1$ | $R_1 + \frac{1}{C_1 s}$ | $a_1 + b_1 \frac{1}{s}$ |
| $Z_2$ | $R_2 + \frac{1}{C_2 s}$ | $a_2 + b_2 \frac{1}{s}$ |
| $Z_3$ | $R_3 + \frac{1}{C_3 s}$ | $a_3 + b_3 \frac{1}{s}$ |
| $Z_4$ | $R_4 + \frac{1}{C_4 s}$ | $a_4 + b_4 \frac{1}{s}$ |

Note that

$$a_i = R_i$$

$$b_i = 1 / C_i$$

The circuit can be further simplified as shown in Figure B-2.

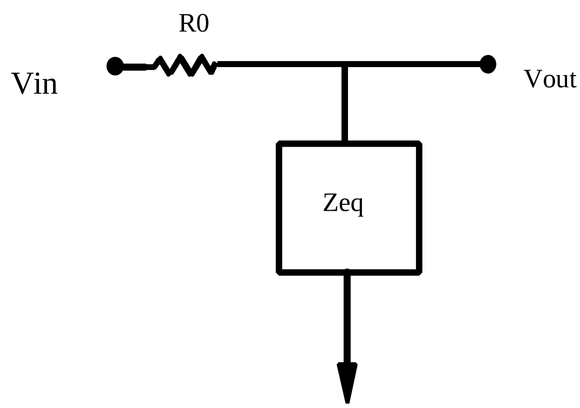


Figure B-2. Equivalent Circuit

The equivalent impedance term is calculated as follows.

$$\frac{1}{Zeq} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \frac{1}{Z_4} \qquad \text{(B-1)}$$

$$Zeq = \frac{Z_1 Z_2 Z_3 Z_4}{Z_1 Z_2 Z_3 + Z_1 Z_2 Z_4 + Z_1 Z_3 Z_4 + Z_2 Z_3 Z_4} \qquad \text{(B-2)}$$

The numerator term is represented as follows.

$$Z_1 Z_2 Z_3 Z_4 = \left[a_1 + \frac{b_1}{s}\right]\left[a_2 + \frac{b_2}{s}\right]\left[a_3 + \frac{b_3}{s}\right]\left[a_4 + \frac{b_4}{s}\right] \qquad \text{(B-3)}$$

$$Z_1 Z_2 Z_3 Z_4 = \left[\frac{1}{s^4}\right][a_1 s + b_1][a_2 s + b_2][a_3 s + b_3][a_4 s + b_4] \qquad \text{(B-4)}$$

$$Z_1 Z_2 Z_3 Z_4 = \left[\frac{1}{s^4}\right]\left[a_1 a_2 s^2 + [a_1 b_2 + a_2 b_1]s + b_1 b_2\right][a_3 s + b_3][a_4 s + b_4] \qquad \text{(B-5)}$$

$$Z_1Z_2Z_3Z_4$$

$$= \left[\frac{1}{s^4}\right]\left[a_1a_2a_3s^3 + [a_1a_2b_3 + a_1a_3b_2 + a_2a_3b_1]s^2 + [a_1b_2b_3 + a_2b_1b_3 + a_3b_1b_2]s + b_1b_2b_3\right]$$
$$[a_4s + b_4]$$

(B-6)

$$Z_1Z_2Z_3Z_4 =$$

$$\left[\frac{1}{s^4}\right]\left[a_1a_2a_3a_4s^4 + [a_1a_2a_3b_4 + a_1a_2a_4b_3 + a_1a_3a_4b_2 + a_2a_3a_4b_1]s^3\right]$$

$$+\left[\frac{1}{s^4}\right]\left[[a_1a_2b_3b_4 + a_1a_3b_2b_4 + a_1a_4b_2b_3 + a_2a_3b_1b_4 + a_2a_4b_1b_3 + a_3a_4b_1b_2]s^2\right]$$

$$+\left[\frac{1}{s^4}\right]\left[[b_1b_2b_3a_4 + b_1b_2b_4a_3 + b_1b_3b_4a_2 + b_2b_3b_4a_1]s + b_1b_2b_3b_4\right]$$

(B-7)

Next, each of the denominator terms is calculated.

$$Z_1Z_2Z_3 = \left[a_1 + \frac{b_1}{s}\right]\left[a_2 + \frac{b_2}{s}\right]\left[a_3 + \frac{b_3}{s}\right] \tag{B-8}$$

$$Z_1Z_2Z_3 = \left[\frac{1}{s^3}\right][a_1s + b_1][a_2s + b_2][a_3s + b_3] \tag{B-9}$$

$$Z_1Z_2Z_3 =$$
$$\left[\frac{1}{s^3}\right]\left[a_1a_2a_3s^3 + [a_1a_2b_3 + a_1a_3b_2 + a_2a_3b_1]s^2 + [a_1b_2b_3 + a_2b_1b_3 + a_3b_1b_2]s + b_1b_2b_3\right]$$

(B-10)

$Z_1Z_2Z_4 =$

$$\left[\frac{1}{s^3}\right]\left[a_1a_2a_4s^3 + \left[a_1a_2b_4 + a_1a_4b_2 + a_2a_4b_1\right]s^2 + \left[a_1b_2b_4 + a_2b_1b_4 + a_4b_1b_2\right]s + b_1b_2b_4\right]$$

(B-11)

$Z_1Z_3Z_4 =$

$$\left[\frac{1}{s^3}\right]\left[a_1a_3a_4s^3 + \left[a_1a_3b_4 + a_1a_4b_3 + a_3a_4b_1\right]s^2 + \left[a_1b_3b_4 + a_3b_1b_4 + a_4b_1b_3\right]s + b_1b_3b_4\right]$$

(B-12)

$Z_2Z_3Z_4 =$

$$\left[\frac{1}{s^3}\right]\left[a_2a_3a_4s^3 + \left[a_2a_3b_4 + a_2a_4b_3 + a_3a_4b_2\right]s^2 + \left[a_2b_3b_4 + a_3b_2b_4 + a_4b_2b_3\right]s + b_2b_3b_4\right]$$

(B-13)

The complete denominator is thus

$$Z_1Z_2Z_3 + Z_1Z_2Z_4 + Z_1Z_3Z_4 + Z_2Z_3Z_4 =$$

$$\begin{aligned}
&\left[\frac{1}{s^3}\right][a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4]s^3 \\
&+\left[\frac{1}{s^3}\right][a_1a_2b_3 + a_1a_3b_2 + a_2a_3b_1 + a_1a_2b_4 + a_1a_4b_2 + a_2a_4b_1]s^2 \\
&+\left[\frac{1}{s^3}\right][a_1a_3b_4 + a_1a_4b_3 + a_3a_4b_1 + a_2a_3b_4 + a_2a_4b_3 + a_3a_4b_2]s^2 \\
&+\left[\frac{1}{s^3}\right][b_1b_2a_3 + b_1b_3a_2 + b_2b_3a_1 + b_1b_2a_4 + b_1b_4a_2 + b_2b_4a_1]s \\
&+\left[\frac{1}{s^3}\right][b_1b_3a_4 + b_1b_4a_3 + b_3b_4a_1 + b_2b_3a_4 + b_2b_4a_3 + b_3b_4a_2]s \\
&+\left[\frac{1}{s^3}\right][b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4]
\end{aligned} \tag{B-14}$$

The voltage transfer function is

$$\frac{Vout}{Vin} = \frac{Zeq}{R_0 + Zeq} \tag{B-15}$$

$$Zeq = \frac{Zn}{Zd} \tag{B-16}$$

$$Zn = Z_1Z_2Z_3Z_4 \tag{B-17}$$

$$Zd = Z_1Z_2Z_3 + Z_1Z_2Z_4 + Z_1Z_3Z_4 + Z_2Z_3Z_4 \tag{B-18}$$

$$\frac{Vout}{Vin} = \frac{\frac{Zn}{Zd}}{R_0 + \frac{Zn}{Zd}} \tag{B-19}$$

$$\frac{Vout}{Vin} = \frac{Zn}{R_0 Zd + Zn} \tag{B-20}$$

$$\frac{Vout}{Vin} = \frac{Z_1 Z_2 Z_3 Z_4}{R_0 \left[Z_1 Z_2 Z_3 + Z_1 Z_2 Z_4 + Z_1 Z_3 Z_4 + Z_2 Z_3 Z_4\right] + Z_1 Z_2 Z_3 Z_4} \tag{B-21}$$

Multiply the left-hand side by $\frac{s^4}{s^4}$.

$$\frac{Vout}{Vin} = \frac{Z_1 Z_2 Z_3 Z_4 s^4}{R_0 \left[Z_1 Z_2 Z_3 + Z_1 Z_2 Z_4 + Z_1 Z_3 Z_4 + Z_2 Z_3 Z_4\right] s^4 + Z_1 Z_2 Z_3 Z_4 s^4} \tag{B-22}$$

$$\frac{Vout}{Vin} = \frac{Vn}{Vd} \tag{B-23}$$

$$Vn = Z_1 Z_2 Z_3 Z_4 s^4 \tag{B-24}$$

$$Vn =$$

$$\left[a_1a_2a_3a_4s^4 + [a_1a_2a_3b_4 + a_1a_2a_4b_3 + a_1a_3a_4b_2 + a_2a_3a_4b_1]s^3\right]$$

$$+\left[[a_1a_2b_3b_4 + a_1a_3b_2b_4 + a_1a_4b_2b_3 + a_2a_3b_1b_4 + a_2a_4b_1b_3 + a_3a_4b_1b_2]s^2\right]$$

$$+[[b_1b_2b_3a_4 + b_1b_2b_4a_3 + b_1b_3b_4a_2 + b_2b_3b_4a_1]s + b_1b_2b_3b_4] \tag{B-25}$$

$$Vn = A_ns^4 + B_ns^3 + C_ns^2 + D_ns + E_n \tag{B-26}$$

$$A_n = a_1a_2a_3a_4 \tag{B-27}$$

$$B_n = [a_1a_2a_3b_4 + a_1a_2a_4b_3 + a_1a_3a_4b_2 + a_2a_3a_4b_1] \tag{B-28}$$

$$C_n = [a_1a_2b_3b_4 + a_1a_3b_2b_4 + a_1a_4b_2b_3 + a_2a_3b_1b_4 + a_2a_4b_1b_3 + a_3a_4b_1b_2] \tag{B-29}$$

$$D_n = [b_1b_2b_3a_4 + b_1b_2b_4a_3 + b_1b_3b_4a_2 + b_2b_3b_4a_1] \tag{B-30}$$

$$E_n = b_1b_2b_3b_4 \tag{B-31}$$

$$Vd = R_1[Z_1Z_2Z_3 + Z_1Z_2Z_4 + Z_1Z_3Z_4 + Z_2Z_3Z_4]s^4 + Z_1Z_2Z_3Z_4s^4 \tag{B-32}$$

$Vd =$

$$\left[a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4\right]R_0\ s^4$$

$$+\left[a_1a_2b_3 + a_1a_3b_2 + a_2a_3b_1 + a_1a_2b_4 + a_1a_4b_2 + a_2a_4b_1\right]R_0\,s^3$$
$$+\left[a_1a_3b_4 + a_1a_4b_3 + a_3a_4b_1 + a_2a_3b_4 + a_2a_4b_3 + a_3a_4b_2\right]R_0\,s^3$$

$$+\left[b_1b_2a_3 + b_1b_3a_2 + b_2b_3a_1 + b_1b_2a_4 + b_1b_4a_2 + b_2b_4a_1\right]R_0\ s^2$$
$$+\left[b_1b_3a_4 + b_1b_4a_3 + b_3b_4a_1 + b_2b_3a_4 + b_2b_4a_3 + b_3b_4a_2\right]R_0\ s^2$$

$$+\left[b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4\right]R_0\,s$$

$$+\left[a_1a_2a_3a_4s^4 + \left[a_1a_2a_3b_4 + a_1a_2a_4b_3 + a_1a_3a_4b_2 + a_2a_3a_4b_1\right]s^3\right]$$

$$+\left[\left[a_1a_2b_3b_4 + a_1a_3b_2b_4 + a_1a_4b_2b_3 + a_2a_3b_1b_4 + a_2a_4b_1b_3 + a_3a_4b_1b_2\right]s^2\right]$$

$$+\left[\left[b_1b_2b_3a_4 + b_1b_2b_4a_3 + b_1b_3b_4a_2 + b_2b_3b_4a_1\right]s + b_1b_2b_3b_4\right]$$

(B-33)

$$Vd = A_ds^4 + B_ds^3 + C_ds^2 + D_ds + E_d \tag{B-34}$$

$$A_d = a_1a_2a_3a_4 + \left[a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4\right]R_0 \tag{B-35}$$

$B_d =$

$$[a_1a_2a_3b_4 + a_1a_2a_4b_3 + a_1a_3a_4b_2 + a_2a_3a_4b_1]$$

$$+[a_1a_2b_3 + a_1a_3b_2 + a_2a_3b_1 + a_1a_2b_4 + a_1a_4b_2 + a_2a_4b_1 +]R_0$$

$$+[a_1a_3b_4 + a_1a_4b_3 + a_3a_4b_1 + a_2a_3b_4 + a_2a_4b_3 + a_3a_4b_2]R_0$$

(B-36)

$C_d =$

$$[a_1a_2b_3b_4 + a_1a_3b_2b_4 + a_1a_4b_2b_3 + a_2a_3b_1b_4 + a_2a_4b_1b_3 + a_3a_4b_1b_2]$$

$$+[b_1b_2a_3 + b_1b_3a_2 + b_2b_3a_1 + b_1b_2a_4 + b_1b_4a_2 + b_2b_4a_1]R_0$$

$$+[b_1b_3a_4 + b_1b_4a_3 + b_3b_4a_1 + b_2b_3a_4 + b_2b_4a_3 + b_3b_4a_2]R_0$$

(B-37)

$D_d =$

$$+[b_1b_2b_3a_4 + b_1b_2b_4a_3 + b_1b_3b_4a_2 + b_2b_3b_4a_1]$$

$$[b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4]R_0$$

(B-38)

$$E_d = b_1b_2b_3b_4 \qquad \text{(B-39)}$$

The values from Tables 1 and B-1 are given in Table B-2 for the resistors and in Table B-3 for the capacitors.

| Table B-2. Resistor Terms | |
|---|---|
| Term | Value (Ω) |
| R0 | 5600 |
| a1 | 2700 |
| a2 | 820 |
| a3 | 270 |
| a4 | 0 |

| Table B-3. Inverse Capacitor Terms | | |
|---|---|---|
| Term | Value (1/F) | Equivalent Value (1/F) |
| B1 | 1/[1e-06] | 1e+06 |
| B2 | 1/[0.316e-06] | 3.165e+06 |
| B3 | 1/[0.1e-06] | 1e+07 |
| B4 | 1/[0.05e-06] | 2e+07 |

The constant terms are given in Table B-4.

| Table B-4. Constant Terms | |
|---|---|
| Term | Value |
| An | 0 |
| Bn | 1.19556e+016 |
| Cn | 4.933672e+020 |
| Dn | 1.889949e+024 |
| En | 6.329114e+026 |
| Ad | 3.347568e+012 |
| Bd | 5.045112e+017 |
| Cd | 6.139691e+021 |
| Dd | 7.085899e+024 |
| Ed | 6.329114e+026 |

The transfer function is

$$\frac{\text{Vout}}{\text{Vin}} = \frac{A_n s^4 + B_n s^3 + C_n s^2 + D_n s + E_n}{A_d s^4 + B_d s^3 + C_d s^2 + D_d s + E_d} \qquad \text{(B-40)}$$

Since An=0, the transfer function can be simplified as

$$\frac{Vout}{Vin} = \frac{B_n s^3 + C_n s^2 + D_n s + E_n}{A_d s^4 + B_d s^3 + C_d s^2 + D_d s + E_d} \tag{B-41}$$

The polynomial terms can be scaled as follows.

$$\frac{Vout}{Vin} = \left\{\frac{B_n}{A_d}\right\}\left\{\frac{s^3 + \left[\frac{C_n}{B_n}\right]s^2 + \left[\frac{D_n}{B_n}\right]s + \left[\frac{E_n}{B_n}\right]}{s^4 + \left[\frac{B_d}{A_d}\right]s^3 + \left[\frac{C_d}{A_d}\right]s^2 + \left[\frac{D_d}{A_d}\right]s + \left[\frac{E_d}{A_d}\right]}\right\} \tag{B-42}$$

$$\frac{Vout}{Vin} = \left\{\hat{A}_n\right\}\left\{\frac{s^3 + \hat{C}_n s^2 + \hat{D}_n s + \hat{E}_n}{s^4 + \hat{B}_d s^3 + \hat{C}_d s^2 + \hat{D}_d s + \hat{E}_d}\right\} \tag{B-43}$$

The scaled constant terms are given in Table B-5.

| Table B-5. Scaled Constant Terms | |
|---|---|
| Formula | Value |
| $\hat{A}_n = \frac{B_n}{A_d}$ | 3571.429 |
| $\hat{C}_n = \frac{C_n}{B_n}$ | 41266.62 |
| $\hat{D}_n = \frac{D_n}{B_n}$ | 1.580807e+008 |
| $\hat{E}_n = \frac{E_n}{B_n}$ | 5.293849e+010 |
| $\hat{B}_d = \frac{B_d}{A_d}$ | 150709.8 |
| $\hat{C}_d = \frac{C_d}{A_d}$ | 1.834075e+009 |
| $\hat{D}_d = \frac{D_d}{A_d}$ | 2.11673e+012 |
| $\hat{E}_d = \frac{E_d}{A_d}$ | 1.89066e+014 |

Convert the transfer function to the following form.

$$\frac{Vout}{Vin} = \left\{\hat{A}_n\right\}\left\{\frac{(s-\alpha_1)(s-\alpha_2)(s-\alpha_3)}{(s-\beta_1)(s-\beta_2)(s-\beta_3)(s-\beta_4)}\right\} \tag{B-52}$$

The alpha terms are the poles. The beta terms are the zeros. Each beta term must be less than zero to ensure stability.

For simplicity, set $\hat{A}_n = 1$ in equation (B-52). The output signal can be scaled to give a desired standard deviation by applying the appropriate scale factor.

The poles and zeros are found via the Newton-Raphson method. The results are shown in Table B-6.

| Table B-6. Poles and Zeros | |
|---|---|
| Term | Value |
| $\alpha_1$ | -.370370E+03 |
| $\alpha_2$ | -.385922E+04 |
| $\alpha_3$ | -.370370E+05 |
| $\beta_1$ | -.974889E+02 |
| $\beta_2$ | -.118037E+04 |
| $\beta_3$ | -.119508E+05 |
| $\beta_4$ | -.137481E+06 |

The polynomial is expanded into partial fractions.

$$\left\{\frac{(s-\alpha_1)(s-\alpha_2)(s-\alpha_3)}{(s-\beta_1)(s-\beta_2)(s-\beta_3)(s-\beta_4)}\right\} = \frac{\lambda_1}{(s-\beta_1)} + \frac{\lambda_2}{(s-\beta_2)} + \frac{\lambda_3}{(s-\beta_3)} + \frac{\lambda_4}{(s-\beta_4)} \qquad \text{(B-53)}$$

$$\begin{aligned}(s-\alpha_1)(s-\alpha_2)(s-\alpha_3) = \\ &+\lambda_1(s-\beta_2)(s-\beta_3)(s-\beta_4) \\ &+\lambda_2(s-\beta_1)(s-\beta_3)(s-\beta_4) \\ &+\lambda_3(s-\beta_1)(s-\beta_2)(s-\beta_4) \\ &+\lambda_4(s-\beta_1)(s-\beta_2)(s-\beta_3)\end{aligned}$$

(B-54)

$$\begin{aligned}(s-\alpha_1)\left(s^2-(\alpha_2+\alpha_3)s+\alpha_2\alpha_3\right) = \\ &+\lambda_1(s-\beta_2)\left(s^2-(\beta_3+\beta_4)s+\beta_3\beta_4\right) \\ &+\lambda_2(s-\beta_1)\left(s^2-(\beta_3+\beta_4)s+\beta_3\beta_4\right) \\ &+\lambda_3(s-\beta_1)\left(s^2-(\beta_2+\beta_4)s+\beta_2\beta_4\right) \\ &+\lambda_4(s-\beta_1)\left(s^2-(\beta_2+\beta_3)s+\beta_2\beta_3\right)\end{aligned}$$

(B-55)

$$s^3-(\alpha_1+\alpha_2+\alpha_3)s^2+(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)s-\alpha_1\alpha_2\alpha_3=$$

$$+\lambda_1\left[s^3-(\beta_2+\beta_3+\beta_4)s^2+(\beta_2\beta_3+\beta_3\beta_4+\beta_2\beta_4)s-\beta_2\beta_3\beta_4\right]$$

$$+\lambda_2\left[s^3-(\beta_1+\beta_3+\beta_4)s^2+(\beta_1\beta_3+\beta_1\beta_4+\beta_3\beta_4)s-\beta_1\beta_3\beta_4\right]$$

$$+\lambda_3\left[s^3-(\beta_1+\beta_2+\beta_4)s^2+(\beta_1\beta_2+\beta_2\beta_4+\beta_1\beta_4)s-\beta_1\beta_2\beta_4\right]$$

$$+\lambda_4\left[s^3-(\beta_1+\beta_2+\beta_3)s^2+(\beta_1\beta_2+\beta_2\beta_3+\beta_1\beta_3)s-\beta_1\beta_2\beta_3\right]$$

(B-56)

$$s^3-(\alpha_1+\alpha_2+\alpha_3)s^2+(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)s-\alpha_1\alpha_2\alpha_3=$$

$$+[\lambda_1+\lambda_2+\lambda_3+\lambda_4]s^3$$

$$-[\lambda_1(\beta_2+\beta_3+\beta_4)+\lambda_2(\beta_1+\beta_3+\beta_4)+\lambda_3(\beta_1+\beta_2+\beta_4)+\lambda_4(\beta_1+\beta_2+\beta_3)]\,s^2$$

$$+[\lambda_1(\beta_2\beta_3+\beta_3\beta_4+\beta_2\beta_4)+\lambda_2(\beta_1\beta_3+\beta_1\beta_4+\beta_3\beta_4)]\,s$$
$$+[\lambda_3(\beta_1\beta_2+\beta_2\beta_4+\beta_1\beta_4)+\lambda_4(\beta_1\beta_2+\beta_2\beta_3+\beta_1\beta_3)]\,s$$

$$-[\lambda_1\beta_2\beta_3\beta_4+\lambda_2\beta_1\beta_3\beta_4+\lambda_3\beta_1\beta_2\beta_4+\lambda_4\beta_1\beta_2\beta_3]$$

(B-57)

$$\lambda_1+\lambda_2+\lambda_3+\lambda_4=1 \qquad \text{(B-58)}$$

$$\lambda_1(\beta_2+\beta_3+\beta_4)+\lambda_2(\beta_1+\beta_3+\beta_4)+\lambda_3(\beta_1+\beta_2+\beta_4)+\lambda_4(\beta_1+\beta_2+\beta_3)=\alpha_1+\alpha_2+\alpha_3$$

(B-59)

$$
\begin{aligned}
&[\lambda_1(\beta_2\beta_3+\beta_3\beta_4+\beta_2\beta_4)+\lambda_2(\beta_1\beta_3+\beta_1\beta_4+\beta_3\beta_4)] \\
&\quad +[\lambda_3(\beta_1\beta_2+\beta_2\beta_4+\beta_1\beta_4)+\lambda_4(\beta_1\beta_2+\beta_2\beta_3+\beta_1\beta_3)]=\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3
\end{aligned}
\tag{B-60}
$$

$$
[\lambda_1\beta_2\beta_3\beta_4+\lambda_2\beta_1\beta_3\beta_4+\lambda_3\beta_1\beta_2\beta_4+\lambda_4\beta_1\beta_2\beta_3]=\alpha_1\alpha_2\alpha_3 \tag{B-61}
$$

Equations B-58 through B-60 can be solved through symbolic algebraic manipulation. Nevertheless, a direct numerical substitution approach is used in this analysis. The lambda terms are then found via Gaussian elimination. The results are shown in Table B-7.

| Table B-7. | |
|---|---|
| Term | Value |
| $\lambda_1$ | 2.1502912E-02 |
| $\lambda_2$ | 4.8943068E-02 |
| $\lambda_3$ | 1.4668036E-01 |
| $\lambda_4$ | 7.8287367E-01 |

The partial fraction expansion is thus complete.

Now consider a generic transfer function.

$$
H_g(s)=\frac{\lambda}{(s-\beta)} \tag{B-62}
$$

The impulse response function is

$$
h_g(t)=\lambda\exp(\beta t) \tag{B-63}
$$

Again, $\beta < 0$ for stability.

The Z-transform from Reference 4 is

$$
H_g(z)=\lambda\left[\frac{z}{z-\exp(\beta T)}\right] \tag{B-64}
$$

T is the time increment.

The transfer function can be represented by a series of $\hat{a}_n$ and $\hat{b}_n$ coefficients as follows

$$H_g(z) = \frac{\hat{b}_0 + \hat{b}_1 z^{-1} + \cdots + \hat{b}_L z^{-L}}{1 + \hat{a}_1 z^{-1} + \cdots + \hat{a}_L z^{-L}} \tag{B-65}$$

The coefficients are constants which determine the system response.  Note the H(z) defines the direct form transfer function for an Lth-order, linear, time-invariant digital system.

Reference 3 gives the time domain equivalent of H(z) as

$$y_k = \left\{ \sum_{n=0}^{L} \hat{b}_n x_{k-n} \right\} - \left\{ \sum_{n=1}^{L} \hat{a}_n y_{k-n} \right\} \tag{B-66}$$

Let L=1.  The first-order, generic transfer function is

$$H_g(z) = \frac{\hat{b}_0 + \hat{b}_1 z^{-1}}{1 + \hat{a}_1 z^{-1}} \tag{B-67}$$

Multiply through by z.

$$H_g(z) = \frac{\hat{b}_0 z + \hat{b}_1}{z + \hat{a}_1} \tag{B-68}$$

A comparison of equations (B-68) and (B-64) yields

$$\hat{b}_0 = \lambda \tag{B-69}$$

$$\hat{b}_1 = 0 \tag{B-70}$$

$$\hat{a}_1 = -\exp(\beta T) \tag{B-71}$$

$$H_g(z) = \frac{\lambda z}{z - \exp(\beta T)} \tag{B-72}$$

The first-order form of equation (B-66) is

$$y_k = b_0 x_k + b_1 x_{k-1} - a_1 y_{k-1} \tag{B-73}$$

By substitution,

$$y_k = \lambda x_k + \exp(\beta T) y_{k-1} \tag{B-74}$$

The transfer function in equation (B-53) can now be applied in the form of a parallel connection of four filters, as shown in Figure B-1. Note that a similar diagram is given in Reference 5.

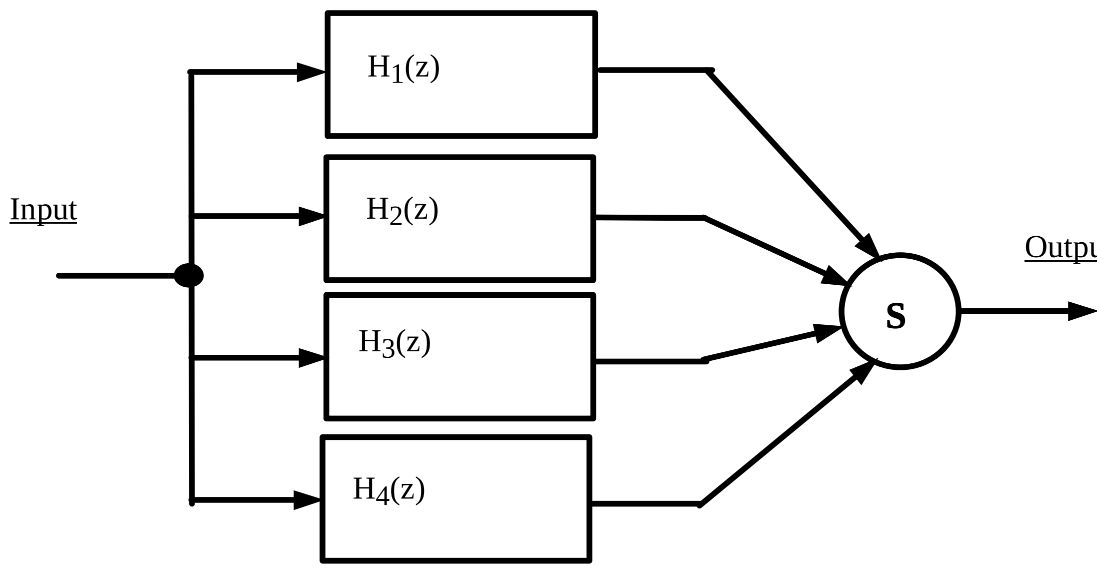


Figure B-1.

The individual transfer functions are

$$H_1(z) = \frac{\lambda_1 z}{z - \exp(\beta_1 T)} \tag{B-72}$$

$$H_2(z) = \frac{\lambda_2 z}{z - \exp(\beta_2 T)} \tag{B-72}$$

$$H_3(z) = \frac{\lambda_3 z}{z - \exp(\beta_3 T)} \tag{B-72}$$

$$H_4(z) = \frac{\lambda_4 z}{z - \exp(\beta_4 T)} \tag{B-72}$$

The corresponding recursive equations are

$$y_{1,j} = \lambda_1 x_j + \exp(\beta_1 T) y_{j-1} \quad \text{(B-74)}$$

$$y_{2,j} = \lambda_2 x_j + \exp(\beta_2 T) y_{j-1} \quad \text{(B-74)}$$

$$y_{3,j} = \lambda_3 x_j + \exp(\beta_3 T) y_{j-1} \quad \text{(B-74)}$$

$$y_{4,j} = \lambda_4 x_j + \exp(\beta_4 T) y_{j-1} \quad \text{(B-74)}$$

Recall that the beta terms are given in Table B-6 and that the lambda terms are given in Table B-7.

The final time series is summed after the completion of the individual recursive equations.

$$y_j = y_{1,j} + y_{2,j} + y_{3,j} + y_{4,j} \quad \text{(B-75)}$$

A practical difficulty, however, is that a sample rate of 1 million samples per second is required for a reasonable accurate 3 dB/octave roll-off. Recall that this approach was based on an analog circuit. The high sample rate is required to force the digital signal to behave as an approximate analog signal.

Example

A white noise signal was generated with a duration of 2 seconds and a sampling rate of 1 million samples per seconds. The white noise signal was then passed through the digital filter shown in equation (B-75).

The resulting pink noise signal is shown along with the white noise signal in Figure B-2. Note that only 0.05 seconds are shown. Also, note that each signal is scaled to have a standard deviation of 1.

The power spectral density of the pink noise signal is shown in Figure B-3. The calculation was made for the entire 2 seconds. The roll-off is approximately 3 dB/octave.

The power spectrum of the pink noise signal as a function of one-third octave bands is given in Figure B-4. The amplitude remains reasonably constant, as expected. The data is shown in terms of bars, although it could have been represented in terms of a piecewise continuous curve.

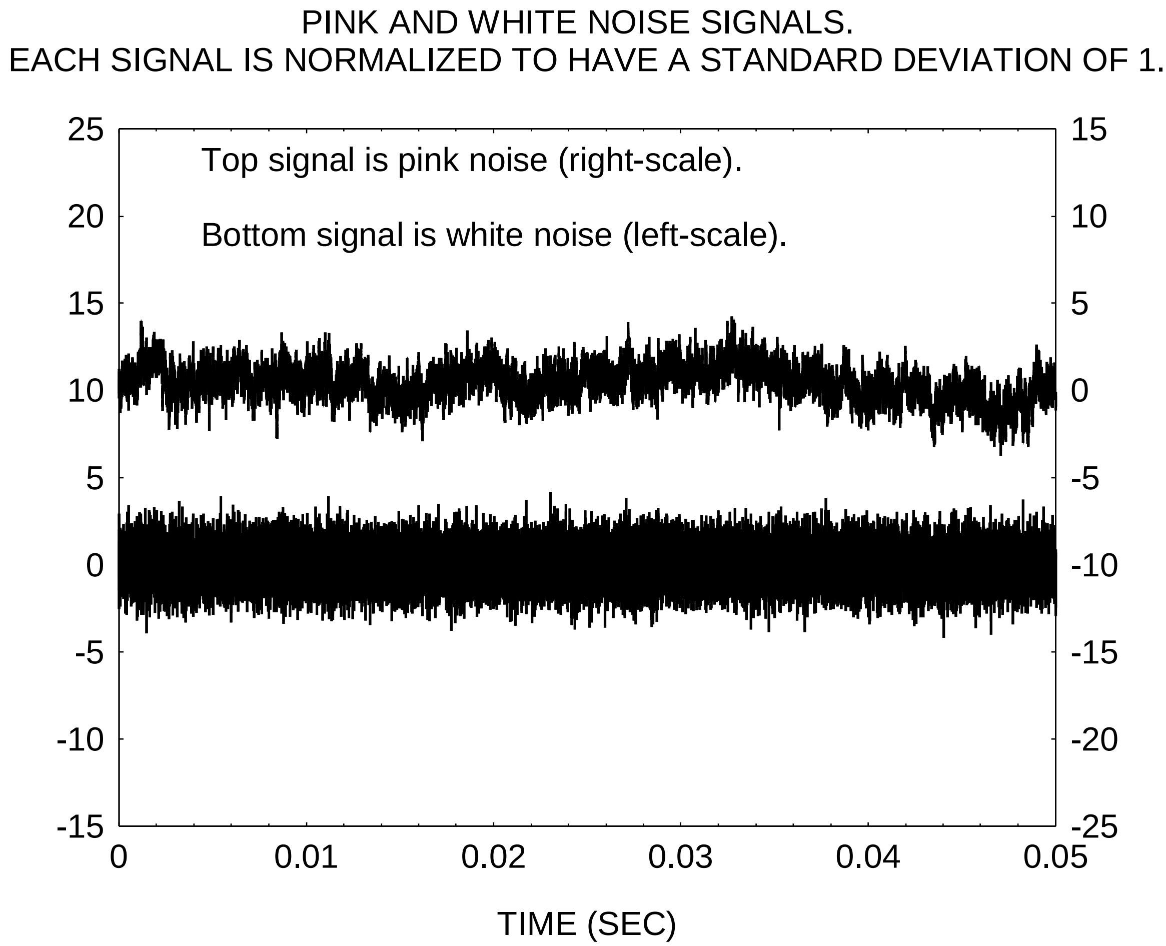
PINK AND WHITE NOISE SIGNALS.
EACH SIGNAL IS NORMALIZED TO HAVE A STANDARD DEVIATION OF 1.
Top signal is pink noise (right-scale).
Bottom signal is white noise (left-scale).
25
20
15
10
5
0
-5
-10
-15
15
10
5
0
-5
-10
-15
-20
-25
0
0.01
0.02
0.03
0.04
0.05
TIME (SEC)

Figure B-2.

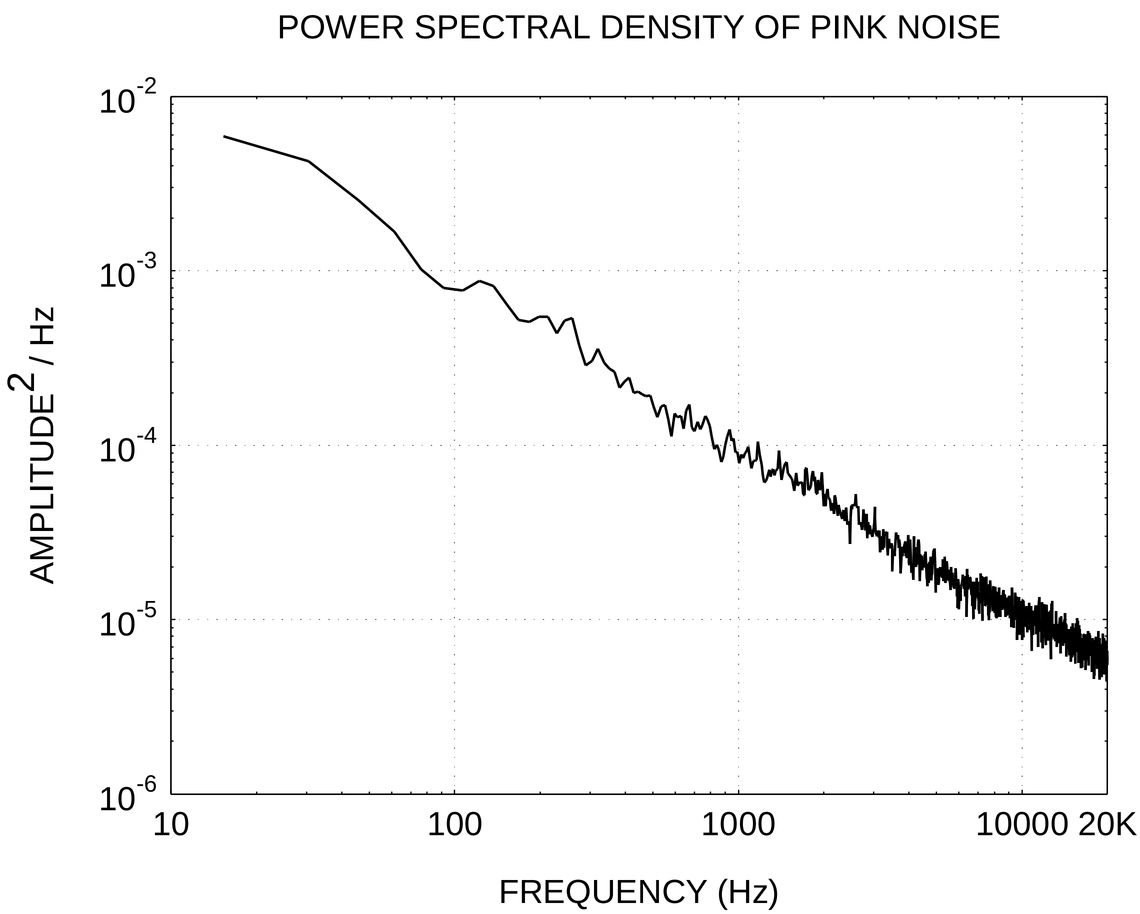
POWER SPECTRAL DENSITY OF PINK NOISE
10^-2
10^-3
10^-4
10^-5
10^-6
AMPLITUDE^2 / Hz
10
100
1000
10000 20K
FREQUENCY (Hz)

Figure B-3.

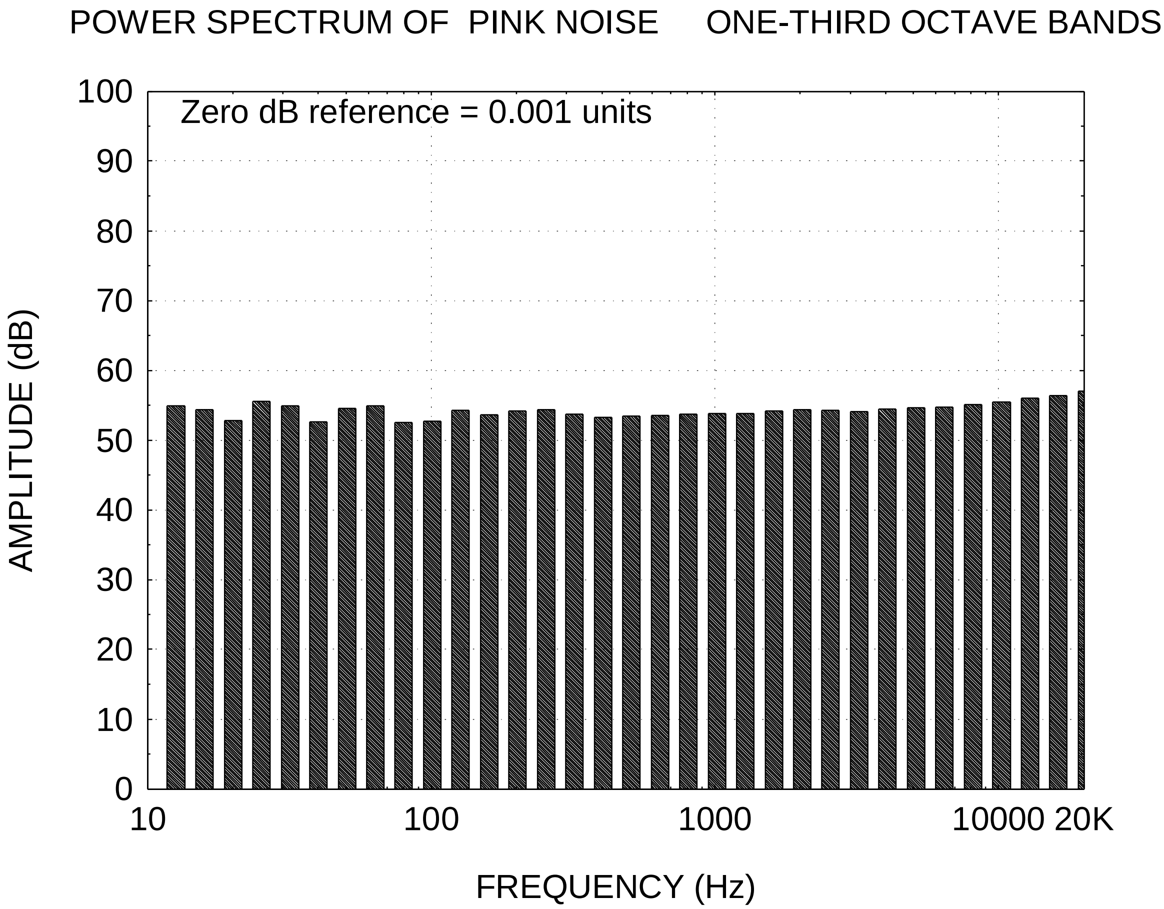
POWER SPECTRUM OF PINK NOISE ONE-THIRD OCTAVE BANDS
Zero dB reference = 0.001 units
AMPLITUDE (dB)
100
90
80
70
60
50
40
30
20
10
0
10
100
1000
10000 20K
FREQUENCY (Hz)

Figure B-4.

# APPENDIX C

The following program generates a white noise signal. The white noise has a Gaussian probability density function.

```
#include <math.h>
#include <string.h>
#include <stdio.h>
#include <stdlib.h>

#define MAX 1200000

void gauss(void);

long i,j,np;

double a[MAX],aq,ave,delta,dt,ms,sdg,sr,ss,stddev,t,tmax,var,x,y,z;
double b[1000],sum[1000];

const double pi=atan2(0.,-1.);
const double e=1./sqrt(2.*pi);
const double max=32767.;

char filename[2][16];
FILE *pFile[2];

void main()
{
     printf("\n ");
     printf("\n white.cpp \n");
     printf("\n by Tom Irvine");
     printf("\n Email: tomirvine@aol.com ");
     printf("\n ");
     printf("\n This program generates a white noise time history.
\n");

       strcpy(filename[0],"a.out");
         pFile[0]=fopen(filename[0],"w");

       printf("\n Enter sampling rate (sample/seconds): \n");
       scanf("%lf",&sr);

       dt=1./sr;

       printf("\n Enter duration (sec): \n");
       scanf("%lf",&tmax);

         printf("\n Enter standard deviation: \n");
       scanf("%lf",&sdg);
```

```
        np=long(tmax/dt);

         if(np > MAX){np=MAX;}

        printf("\n np= %ld",np);

         if( np > 2 )
         {
//  The gauss function determines the probability for
//  each Nsigma value.

          gauss();

           ave = 0.;
           ms = 0.;

           for(i=0; i<np; i++)
           {
            x=rand()/max;
            a[i]=0.;

              for(j=1; j<= 599; j++)
              {

//   x is a probability value.  0 < x < 1

              if( x >= sum[j] &&  x <= sum[j+1] )
                  {

//    a = Nsigma value which corresponds to probability x.

                  a[i]=b[j];
                  break;
              }
            }

//     50% of the amplitude points are multiplied by -1.

              y=rand()/max;

              if(y <= 0.5){a[i]*=-1;}

              ave+=a[i];
              ms+=pow(a[i],2.);
           }

           ave/=double(np);
           ms/=double(np);

           var=ms-pow(ave,2.);
           stddev=sqrt(fabs(var));
```

```
//    print the output file

            for(i=0; i<np; i++)
            {
                  t=dt*i;
                a[i]=(a[i]-ave)*(sdg/stddev);
              fprintf(pFile[0],"%14.7e %14.7e \n",t,a[i]);
          }

            printf("\n\n The output file is:  a.out ");
            printf("\n\n This file has two columns:  time(sec) and
amplitude.");
            printf("\n Please call this file into your own plotting
program.\n\n");
        }
        fclose(pFile[0]);
}

void gauss(void)
{
       strcpy(filename[1],"gauss.out");
        pFile[1]=fopen(filename[1],"w");

/*
     b[j] is a standard deviation.  The maximum b[j] is six sigma.
     sum[j] is the area under the Gaussian curve from 0 to b[j].
*/
        double ab;

       for( j=1;j<=600;j++)
        {
            b[j]=j/100.;
            ab=-b[j];
            ss=0.;
            delta=(b[j]-ab)/1000.;

            for( i=0; i<=1000; i++)
                 {
               z=(delta*i)+ab;
               ss+=exp( ( -pow(z,2.) )/2.);
            }

            sum[j]=ss*e*delta;
            aq=1.-sum[j];

            fprintf(pFile[1]," %14.7e %14.7e %14.7e
\n",b[j],sum[j],aq);

    }
    fclose(pFile[1]);
```