ACOUSTICAL NATURAL FREQUENCIES OF PIPES Revision B

By Tom Irvine
Email: tomirvine@aol.com

March 20, 2005

Introduction

The natural frequencies for pipes with constant cross-section are given in Table 1.

Table 1. Natural Frequencies of Pressure Oscillation

Configuration	Frequency (Hz)	Source
Open-Open	$f_n = \frac{n}{2}\frac{c}{L}$	Equation (35b)
Closed-Open	$f_n = \left(\frac{2n-1}{4}\right)\frac{c}{L}$	Equation (54b)
Closed-Closed	$f_n = \frac{n}{2}\frac{c}{L}$	Equation (71b)
Driven by piston at one end. Open at other end. Large flange at open end.	$f_n = \frac{n}{2}\frac{c}{\left[L + \frac{8a}{3\pi}\right]}$	Reference 1
Driven by piston at one end. Open at other end. Unflanged.	$f_n = \frac{n}{2}\frac{c}{\left[L + 0.6\,a\right]}$	Reference 1

where

n = 1, 2, 3, ….

c is the speed of sound

L is the length

a is the radius

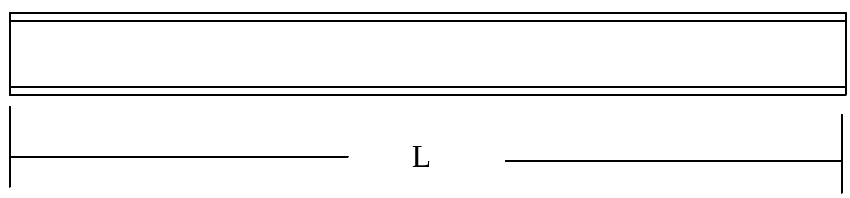
Note that the open-open and closed-closed pipes have the same formula.

Derivation

Acoustic waves require a physical medium through which to propagate. Sound cannot travel in a vacuum. On the other hand, sound can travel through the air, water, Earth, metal, wood, and other physical objects.

An acoustic wave is a longitudinal pressure wave which alternately pushes and pulls the substance through which it propagates. The amplitude disturbance is thus parallel to the direction of propagation.

Consider the pipe in Figure 1, where the length is much greater than the diameter. The cross-section may have an arbitrary shape but must be constant. Assume that the pipe is filled with some gas or liquid.



L is the length

c is the speed of sound in the enclosed gas or liquid

Figure 1.

The acoustic pressure p(x, t) is governed by the equation

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \quad (1)$$

This equation is taken from Reference 1. Note that equation (1) has the same form as the equation for the longitudinal vibration of a rod.

Note that the speed of sound is given by

$$c = \sqrt{\frac{B}{\rho_o}} \quad (2)$$

where

B is the adiabatic bulk modulus

ρ_o is the equilibrium density

The adiabatic bulk modulus B is defined in terms of pressure P and volume V as

$$B = \frac{\Delta P}{-\Delta V / V} \tag{3}$$

The bulk modulus is essentially a measure of stress divided by strain.

Further information about the speed of sound is given in Reference 2.

Separate the variables in equation (1). Let

$$p(x, t) = P(x)T(t) \tag{4}$$

Substitute equation (4) into (1).

$$\frac{\partial^2}{\partial x^2}[P(x)T(t)] = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}[P(x)T(t)] \tag{5}$$

Divide through by P(x)T(t).

$$P''(x)T(t) = \frac{1}{c^2}P(x)T''(t) \tag{6}$$

$$\frac{P''(x)}{P(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} \tag{7}$$

$$c^2 \frac{P''(x)}{P(x)} = \frac{T''(t)}{T(t)} \tag{8}$$

Each side of equation (8) must equal a constant. Let ω be a constant.

$$c^2 \frac{P''(x)}{P(x)} = \frac{T''(t)}{T(t)} = -\omega^2 \tag{9}$$

The time equation is

$$\frac{T''(t)}{T(t)} = -\omega^2 \tag{10}$$

$$T''(t) - \omega^2 T(t) = 0 \tag{11}$$

$$T''(t) + \omega^2\, T(t) \,=\, 0 \tag{12}$$

Propose a solution

$$T(t) = a\sin(\omega t) + b\cos(\omega t) \tag{13}$$

$$T'(t) = a\,\omega\cos(\omega t) - b\,\omega\sin(\omega t) \tag{14}$$

$$T''(t) = -a\,\omega^2\sin(\omega t) - b\,\omega^2\cos(\omega t) \tag{15}$$

Verify the proposed solution. Substitute in equation (12).

$$-a\,\omega^2\sin(\omega t) - b\,\omega^2\cos(\omega t) \quad + \omega^2\left[a\sin(\omega t) + b\cos(\omega t)\right] = 0 \tag{16}$$

$$0 = 0 \tag{17}$$

Equation (13) is thus verified as a solution.

There is not a unique ω, however, in equation (9). This is demonstrated later in the derivation. Thus, a subscript n must be added as follows.

$$T_n(t) = a_n\sin(\omega_n t) + b_n\cos(\omega_n t) \tag{18}$$

The spatial equation is

$$c^2\,\frac{P''(x)}{P(x)} = -\omega^2 \tag{19}$$

$$c^2\,P''(x) = -\omega^2 P(x) \tag{20}$$

$$c^2\,P''(x) + \omega^2 P(x) = 0 \tag{21}$$

$$P''(x) + \frac{\omega^2}{c^2} P(x) = 0 \tag{22}$$

Equation (22) is similar to equation (12). Thus, a solution can be found be inspection.

$$P(x) = d \sin\left(\frac{\omega x}{c}\right) + e \cos\left(\frac{\omega x}{c}\right) \tag{23}$$

$$P'(x) = \left(\frac{\omega}{c}\right)\left[d \cos\left(\frac{\omega x}{c}\right) - e \sin\left(\frac{\omega x}{c}\right)\right] \tag{24}$$

Now consider three boundary condition cases.

Case I. Both Ends Open

The left boundary condition is

$$p(0,t) = 0 \quad \text{(zero acoustic pressure)} \tag{25}$$

$$P(0)T(t) = 0 \tag{26}$$

$$P(0) = 0 \tag{27}$$

The right boundary condition is

$$p(L,t) = 0 \quad \text{(zero acoustic pressure)} \tag{28}$$

$$P(L)T(t) = 0 \tag{29}$$

$$P(L) = 0 \tag{30}$$

Substitute equation (27) into (23).

$$e = 0 \tag{31}$$

$$P(x) = d \sin\left(\frac{\omega x}{c}\right) \tag{32}$$

Substitute equation (30) into (32).

$$d \sin\left(\frac{\omega L}{c}\right) = 0 \tag{33}$$

The constant d must be non-zero for a non-trivial solution. Thus,

$$\frac{\omega_n L}{c} = n\pi, \quad n = 1, 2, 3 \ldots \tag{34}$$

The ω term is given a subscript n because there are multiple roots. The angular natural frequency with dimensions [radians/time] is thus

$$\omega_n = n\pi \frac{c}{L}, \quad n = 1, 2, 3 \ldots \tag{35a}$$

$$f_n = \frac{n\, c}{2\, L}, \quad n = 1, 2, 3 \ldots \tag{35b}$$

The acoustic pressure function in the open-open pipe is

$$P_n(x) = d_n \sin\left(\frac{\omega_n x}{c}\right) \tag{36}$$

$$P_n(x) = d_n \sin\left(\frac{n\pi x}{L}\right) \tag{37}$$

Substitute the natural frequency term into the time equation.

$$T_n(t) = a_n \sin\left(\frac{n\pi c t}{L}\right) + b_n \cos\left(\frac{n\pi c t}{L}\right) \tag{38}$$

The acoustic pressure function is thus

$$p(x,t) = \sum_{n=1}^{\infty} \left[d_n \sin\left(\frac{n\pi x}{L}\right)\right]\left[a_n \sin\left(\frac{n\pi c t}{L}\right) + b_n \cos\left(\frac{n\pi c t}{L}\right)\right] \tag{39}$$

The coefficients can be simplified as follows

$$A_n = d_n \ a_n \tag{40}$$

$$B_n = d_n \ b_n \tag{41}$$

By substitution, the acoustic pressure equation is

$$p(x,t) = \sum_{n=1}^{\infty} \left[\sin\left(\frac{n\pi x}{L}\right)\right] \left[A_n \sin\left(\frac{n\pi c t}{L}\right) + B_n \cos\left(\frac{n\pi c t}{L}\right)\right] \tag{42}$$

<u>Case II. Open-Closed</u>

The left boundary conditions is

$$p(0, t) = 0 \quad \text{(zero acoustic pressure)} \tag{43}$$

$$P(0)T(t) = 0 \tag{44}$$

$$P(0) = 0 \tag{45}$$

The right boundary condition is

$$\left.\frac{\partial}{\partial x} p(x, t)\right|_{x=L} = 0 \qquad \text{(zero pressure slope)} \tag{46}$$

$$P'(L)T(t) = 0 \tag{47}$$

$$P'(L) = 0 \tag{48}$$

Substitute equation (45) into (23).

$$e = 0 \tag{49}$$

Thus, the acoustic pressure equation becomes

$$P(x) = d \sin\left(\frac{\omega x}{c}\right) \tag{50}$$

$$P'(x) = \left(\frac{\omega}{c}\right)\left[d \cos\left(\frac{\omega x}{c}\right)\right] \tag{51}$$

Substitute equation (48) into (50).

$$d \cos\left(\frac{\omega L}{c}\right) = 0 \tag{52}$$

The constant d must be non-zero for a non-trivial solution. Thus,

$$\frac{\omega_n L}{c} = \left(\frac{2n-1}{2}\right)\pi \;= 1, 2, 3 \ldots \tag{53}$$

The ω term is given a subscript n because there are multiple roots.

$$\omega_n = \left(\frac{2n-1}{2}\right)\pi \frac{c}{L}, \quad n = 1, 2, 3 \ldots \tag{54a}$$

$$f_n = \left(\frac{2n-1}{4}\right) \frac{c}{L}, \quad n = 1, 2, 3 \ldots \tag{54b}$$

The acoustic pressure function for the open-closed pipe is

$$P_n(x) = d_n \sin\left(\frac{\omega_n x}{c}\right) \tag{55}$$

$$P_n(x) = d_n \sin\left(\frac{(2n-1)\pi x}{2L}\right) \tag{56}$$

Substitute the natural frequency term into the time equation.

$$T_n(t) = a_n \sin\left(\frac{(2n-1)\pi ct}{2L}\right) + b_n \cos\left(\frac{(2n-1)\pi ct}{2L}\right) \tag{57}$$

The acoustic pressure function is thus

$$p(x,t) = \sum_{n=1}^{\infty}\left[d_n \sin\left(\frac{(2n-1)\pi x}{2L}\right)\right]\left[a_n \sin\left(\frac{(2n-1)\pi ct}{2L}\right) + b_n \cos\left(\frac{(2n-1)\pi ct}{2L}\right)\right] \tag{58}$$

Simplify the coefficients.

$$p(x,t) = \sum_{n=1}^{\infty}\left[\sin\left(\frac{(2n-1)\pi x}{2L}\right)\right]\left[A_n \sin\left(\frac{(2n-1)\pi ct}{2L}\right) + B_n \cos\left(\frac{(2n-1)\pi ct}{2L}\right)\right] \tag{59}$$

Case III. Both Ends Closed

The left boundary conditions is

$$\frac{\partial}{\partial x} p(x,t)\Big|_{x=0} = 0 \quad \text{(zero pressure slope)} \tag{60}$$

$$P'(0)T(t) = 0 \tag{61}$$

$$P'(0) = 0 \tag{62}$$

$$\frac{\partial}{\partial x} p(x,t)\Big|_{x=L} = 0 \quad \text{(zero pressure slope)} \tag{63}$$

$$P'(L)T(t) = 0 \tag{64}$$

$$P'(L) = 0 \tag{65}$$

Apply equation (62) to (23).

$$d = 0 \tag{66}$$

Then

$$P(x) = e \cos\left(\frac{\omega x}{c}\right) \tag{67}$$

The slope equation is

$$P'(x) = -\left(\frac{\omega}{c}\right) e \sin\left(\frac{\omega x}{c}\right) \tag{68}$$

Substitute equation (65) into (68).

$$e \sin\left(\frac{\omega L}{c}\right) = 0 \tag{69}$$

The constant e must be non-zero for a non-trivial solution. Thus,

$$\frac{\omega_n L}{c} = n\pi , \quad n = 1, 2, 3... \tag{70}$$

The ω term is given a subscript n because there are multiple roots.

$$\omega_n = n\pi \frac{c}{L} , \quad n = 1, 2, 3... \tag{71a}$$

$$f_n = \frac{n}{2}\frac{c}{L} , \quad n = 1, 2, 3... \tag{71b}$$

$$P_n(x) = e_n \cos\left(\frac{\omega_n x}{c}\right) \tag{72}$$

$$P_n(x) = e_n \cos\left(\frac{n\pi x}{L}\right) \tag{73}$$

$$T_n(t) = a_n \sin\left(\frac{n\pi c t}{L}\right) + b_n \cos\left(\frac{n\pi c t}{L}\right) \tag{74}$$

$$p(x,t) = \sum_{n=1}^{\infty} \left[e_n \cos\left(\frac{n\pi x}{L}\right) \right] \left[a_n \sin\left(\frac{n\pi c t}{L}\right) + b_n \cos\left(\frac{n\pi c t}{L}\right) \right] \quad (75)$$

Simplify the coefficients.

$$p(x,t) = \sum_{n=1}^{\infty} \left[\cos\left(\frac{n\pi x}{L}\right) \right] \left[A_n \sin\left(\frac{n\pi c t}{L}\right) + B_n \cos\left(\frac{n\pi c t}{L}\right) \right] \quad (76)$$

<u>Example</u>

A solid rocket motor can be modeled as a closed-closed pipe, because the nozzle throat diameter is very small. A simple diagram is shown in Figure 2.

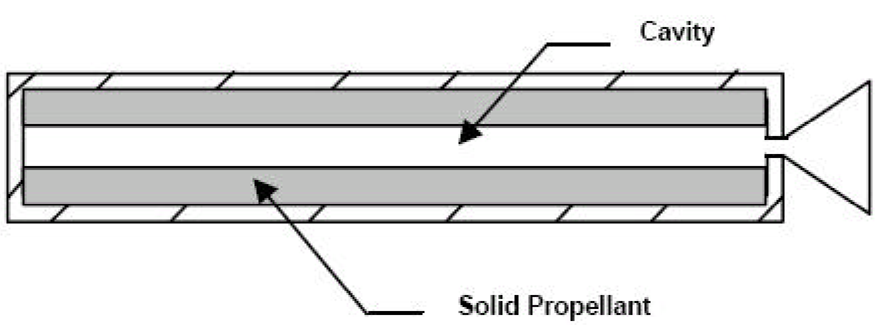


Figure 2.

The speed of sound in the gas inside the motor cavity is typically 3500 feet/sec (1067 meters/sec), due to the high pressure and high temperature. This value is about three times the speed of sound in air at ambient sea level conditions.

A certain rocket motor has an internal cavity length of 27 feet (8.2 meters). Calculate the fundamental acoustical frequency.

The formula for a closed-closed pipe is

$$\omega_n = n\pi \frac{c}{L}, \quad n = 1, 2, 3 \ldots \tag{77}$$

$$\omega_n = 2\pi f_n \tag{78}$$

$$f_n = \frac{n}{2} \frac{c}{L}, \quad n = 1, 2, 3 \ldots \tag{79}$$

The fundamental frequency is

$$f_1 = \frac{1}{2}\left(\frac{c}{L}\right) \tag{80}$$

$$f_1 = \frac{1}{2}\left(\frac{3500 \text{ ft / sec}}{27 \text{ ft}}\right) \tag{81}$$

$$f_1 = 64.8 \text{ Hz} \tag{82}$$

<u>References</u>

1. Lawrence Kinsler et al, Fundamentals of Acoustics, Third Edition, Wiley, New York, 1982.

2. T. Irvine, Formulas for Calculating the Speed of Sound, Vibrationdata, Publications, 2000.