RESPONSE OF A RECTANGULAR PLATE TO BASE EXCITATION
Revision E

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Variables

A_{mn}	amplitude coefficient
a	length
b	width
D	plate stiffness factor
E	elastic modulus
h	plate thickness
k_m	wavenumber
M	plate mass
M_x	bending moment
m, n, u, v	mode number indices
μ	Poisson's ratio
ρ	mass per volume

z	relative displacement, out-of-plane
Z_{mn}	normalized mode shape
Γ_{mn}	participation factor
ω	excitation frequency
ω_{mn}	natural frequency
ξ_{mn}	modal damping
z	relative displacement, out-of-plane
w(t)	base input acceleration
$W(\omega)$	Fourier transform of base input acceleration
$Z(x, y, \omega)$	Relative displacement
$\ddot{U}(x, y, \omega)$	Absolute Acceleration

Introduction

Consider the rectangular plate in Figure 1. Assume that it is simply-supported along each edge.

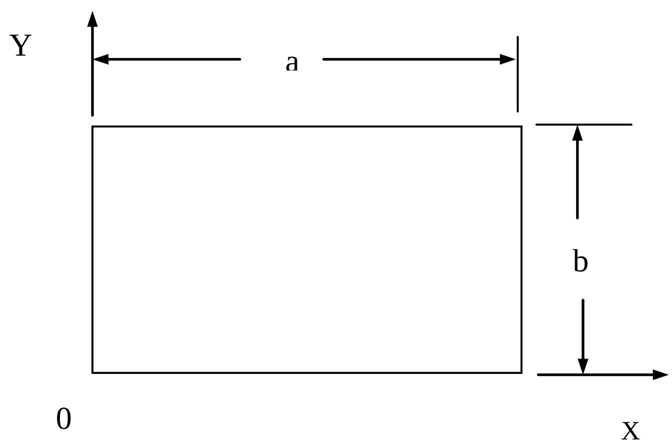

Figure 1.

Normal Modes Analysis

Note that the plate stiffness factor D is given by

$$D = \frac{Eh^3}{12\left(1-\mu^2\right)} \tag{1}$$

The governing equation of motion is

$$D\left(\frac{\partial^4 z}{\partial x^4} + 2\frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4}\right) + \rho h \frac{\partial^2 z}{\partial t^2} = 0 \tag{2}$$

Assume a harmonic response.

$$z(x, y, t) = Z(x, y)\exp(j\omega t) \tag{3}$$

$$D\left(\frac{\partial^4 Z}{\partial x^4}+2\frac{\partial^4 Z}{\partial x^2 \partial y^2}+\frac{\partial^4 Z}{\partial y^4}\right)\exp(j\omega t)-\rho h\omega^2 Z\exp(j\omega t)=0 \tag{4}$$

$$D\left(\frac{\partial^4 Z}{\partial x^4}+2\frac{\partial^4 Z}{\partial x^2 \partial y^2}+\frac{\partial^4 Z}{\partial y^4}\right)-\rho h\omega^2 Z=0 \tag{5}$$

The boundary conditions are

$$Z(x,y)=0, \quad M_x(x,y)=0 \qquad \text{for} \quad x = 0, a \tag{6}$$

$$Z(x,y)=0, \quad M_y(x,y)=0 \qquad \text{for} \quad y = 0, b \tag{7}$$

Assume the following displacement function which satisfies the boundary conditions, where c is an amplitude coefficient and m an n are integers.

$$Z_{mn}(x,y)=A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{8}$$

The partial derivates are

$$\frac{\partial}{\partial x}Z_{mn}=\left(\frac{m\pi}{a}\right)A_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{9}$$

$$\frac{\partial^2}{\partial x^2}Z_{mn}=-\left(\frac{m\pi}{a}\right)^2 A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{10}$$

$$\frac{\partial^3}{\partial x^3}Z_{mn}=-\left(\frac{m\pi}{a}\right)^3 A_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{11}$$

$$\frac{\partial^4}{\partial x^4} Z_{mn} = \left(\frac{m\pi}{a}\right)^4 A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \tag{12}$$

Similarly,

$$\frac{\partial^4}{\partial y^4} Z_{mn} = \left(\frac{n\pi}{n}\right)^4 A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \tag{13}$$

Also,

$$\frac{\partial^2}{\partial x^2 \partial y^2} Z_{mn} = \left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{a}\right)^2 A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \tag{14}$$

$$D\left(\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right) A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) - \rho h \omega^2 A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) = 0 \tag{15}$$

$$D\left(\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right) - \rho h \omega^2 = 0 \tag{16}$$

$$\omega^2 = \frac{D}{\rho h}\left(\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right) \tag{17}$$

$$\omega = \sqrt{\frac{D}{\rho h}\left(\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right)} \tag{18}$$

$$\omega = \sqrt{\frac{D}{\rho h}} \left(\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right) \tag{19}$$

Note that the wave numbers are

$$k_m = m\pi / a \tag{20}$$

$$k_n = n\pi / b \tag{21}$$

<u>Normalized Mode Shapes</u>

The mode shapes are normalized such that

$$\rho h \int_0^b \int_0^a [Z_{mn}(x, y)]^2 \, dx\, dy = 1 \tag{22}$$

$$Z_{mn} = A_{mn} \sin\left(\frac{m\pi x}{a} \right) \sin\left(\frac{n\pi y}{b} \right) \tag{23}$$

$$\rho h \int_0^b \int_0^a \left[A_{mn} \sin\left(\frac{m\pi x}{a} \right) \sin\left(\frac{n\pi y}{b} \right) \right]^2 dx\, dy = 1 \tag{24}$$

$$\rho h \, [A_{mn}]^2 \int_0^b \int_0^a \left[\sin\left(\frac{m\pi x}{a} \right) \sin\left(\frac{n\pi y}{b} \right) \right]^2 dx\, dy = 1 \tag{25}$$

$$\frac{\rho h}{4} [A_{mn}]^2 \int_0^b \int_0^a \left[1 - \cos\left(\frac{2 m\pi x}{a} \right) \right] \left[1 - \cos\left(\frac{2 n\pi y}{b} \right) \right] dx\, dy = 1 \tag{26}$$

$$\frac{\rho h}{4}\left[A_{mn}\right]^2\int_0^b\left[1-\cos\left(\frac{2n\pi y}{b}\right)\right]\left[x-\left(\frac{a}{2m\pi x}\right)\sin\left(\frac{2m\pi x}{a}\right)\right]\Bigg|_0^a dy = 1 \tag{27}$$

$$\frac{\rho h a}{4}\left[A_{mn}\right]^2\int_0^b\left[1-\cos\left(\frac{2n\pi y}{b}\right)\right] dy = 1 \tag{28}$$

$$\frac{\rho h a}{4}\left[A_{mn}\right]^2\left[y-\left(\frac{b}{2n\pi y}\right)\sin\left(\frac{2n\pi y}{b}\right)\right]\Bigg|_0^b = 1 \tag{29}$$

$$\frac{\rho h ab}{4}\left[A_{mn}\right]^2 = 1 \tag{30}$$

$$\left[A_{mn}\right]^2 = \frac{4}{\rho h ab} \tag{31}$$

$$A_{mn} = \sqrt{\frac{4}{\rho h ab}} \tag{32}$$

$$A_{mn} = \frac{2}{\sqrt{\rho h ab}} \tag{33}$$

$$M = \rho h ab \tag{34}$$

$$A_{mn} = \frac{2}{\sqrt{M}} \tag{35}$$

Note that A_{mn} is considered to be dimensionless, although the units must be consistent within the analysis.

Participation Factors

The mass density is constant. Thus

$$\Gamma_{mn} = \rho h \int_0^b \int_0^a Z_{mn}(x,y)\,dx\,dy \tag{36}$$

$$\Gamma_{mn} = \rho h \int_0^b \int_0^a \left[A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)\right] dx\,dy \tag{37}$$

$$\Gamma_{mn} = A_{mn}\, \rho h \int_0^b \int_0^a \left[\sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)\right] dx\,dy \tag{38}$$

$$\Gamma_{mn} = \frac{2\,\rho h}{\sqrt{M}} \int_0^b \int_0^a \left[\sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)\right] dx\,dy \tag{39}$$

$$\Gamma_{mn} = \frac{-2\,\rho h}{\sqrt{M}} \int_0^b \left[\frac{a}{m\pi}\right] \left[\cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)\right]\Bigg|_0^a dy \tag{40}$$

$$\Gamma_{mn} = \frac{-2\,\rho h}{\sqrt{M}} \left[\frac{a}{m\pi}\right] [\cos(m\pi) - 1] \int_0^b \left[\sin\left(\frac{n\pi y}{b}\right)\right] dy \tag{41}$$

$$\Gamma_{mn} = \frac{2\,\rho h}{\sqrt{M}} \left[\frac{a}{m\pi}\right] [\cos(m\pi) - 1] \left[\left[\frac{b}{n\pi}\right]\left[\cos\left(\frac{n\pi x}{b}\right)\right]\right]\Bigg|_0^b \tag{42}$$

$$\Gamma_{mn} = \frac{2\,\rho h}{\sqrt{M}} \left[\frac{ab}{m n \pi^2}\right] [\cos(m\pi) - 1][\cos(n\pi) - 1] \tag{43}$$

$$\Gamma_{mn} = \frac{2\,M}{\sqrt{M}}\left[\frac{1}{mn\pi^2}\right][\cos(m\pi)-1][\cos(n\pi)-1] \tag{44}$$

$$\Gamma_{mn} = 2\sqrt{M}\left[\frac{1}{mn\pi^2}\right][\cos(m\pi)-1][\cos(n\pi)-1] \tag{45}$$

Effective Modal Mass

$$m_{eff,mn} = \frac{\left[\rho h \int_0^b \int_0^a Z_{mn}(x,y)\,dx\,dy\right]^2}{\left[\rho h \int_0^b \int_0^a [Z_{mn}(x,y)\,dx\,dy]^2\right]} \tag{46}$$

The modes shapes are normalized such that

$$\rho h \int_0^b \int_0^a [Z_{mn}(x,y)\,dx\,dy]^2 = 1 \tag{47}$$

Thus

$$m_{eff,mn} = \left[\rho h \int_0^b \int_0^a Z_{mn}(x,y)\,dx\,dy\right]^2 \tag{48}$$

$$m_{eff,mn} = \left[\Gamma_{mn}\right]^2 \tag{49}$$

Response to Base Excitation

The forced response equation for a plate with base motion is

$$D\left(\frac{\partial^4 z}{\partial x^4}+2\frac{\partial^4 z}{\partial x^2 \partial y^2}+\frac{\partial^4 z}{\partial y^4}\right)+\rho h\frac{\partial^2 z}{\partial t^2}=-\rho h\frac{\partial^2 w}{\partial t^2} \tag{50}$$

where w is base excitation.

The term on the right-hand-side is the inertial force per unit area.

The displacement is

$$z(x,y,t)=\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{mn}(x,y)\, T_{mn}(t) \tag{51}$$

$$Z_{mn}(x,y)=\frac{2}{\sqrt{M}}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{52}$$

$$D\left(\frac{\partial^4 z}{\partial x^4}+2\frac{\partial^4 z}{\partial x^2 \partial y^2}+\frac{\partial^4 z}{\partial y^4}\right)+\rho h\frac{\partial^2 z}{\partial t^2}=-\rho h\frac{\partial^2 w}{\partial t^2} \tag{53}$$

$$
D\frac{\partial^4}{\partial x^4}\left[\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{mn}(x,y)\,T_{mn}(t)\right]
$$

$$
+\,2D\frac{\partial^4}{\partial x^2\partial y^2}\left[\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{mn}(x,y)\,T_{mn}(t)\right]
$$

$$
+\,D\frac{\partial^4}{\partial y^4}\left[\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{mn}(x,y)\,T_{mn}(t)\right]
$$

$$
+\,\rho h\frac{\partial^2}{\partial t^2}\left[\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{mn}(x,y)\,T_{mn}(t)\right]=-\rho h\frac{\partial^2 w}{\partial t^2} \tag{54}
$$

$$D\left[\sum_{m=1}^{q}\sum_{n=1}^{r}\frac{\partial^4}{\partial x^4}Z_{mn}(x,y)\,T_{mn}(t)\right]$$

$$+2D\left[\sum_{m=1}^{q}\sum_{n=1}^{r}\frac{\partial^4}{\partial x^2\partial y^2}Z_{mn}(x,y)\,T_{mn}(t)\right]$$

$$+D\left[\sum_{m=1}^{q}\sum_{n=1}^{r}\frac{\partial^4}{\partial y^4}Z_{mn}(x,y)\,T_{mn}(t)\right]$$

$$+\rho h\left[\sum_{m=1}^{q}\sum_{n=1}^{r}Z_{mn}(x,y)\,\frac{\partial^2}{\partial t^2}T_{mn}(t)\right]=-\rho h\frac{\partial^2 w}{\partial t^2} \tag{55}$$

$$Z_{mn}(x,y)=\frac{2}{\sqrt{M}}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{56}$$

$$\frac{\partial^4}{\partial x^4}Z_{mn}(x,y)=\frac{\partial^4}{\partial x^4}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{57}$$

$$\frac{\partial^4}{\partial x^4}Z_{mn}(x,y)=\left(\frac{m\pi}{a}\right)^4 Z_{mn}(x,y) \tag{58}$$

$$\frac{\partial^4}{\partial x^2\partial y^2}Z_{mn}(x,y)=\frac{\partial^4}{\partial x^2\partial y^2}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{59}$$

$$\frac{\partial^4}{\partial x^2 \partial y^2} Z_{mn}(x,y) = \left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{60}$$

$$\frac{\partial^4}{\partial x^2 \partial y^2} Z_{mn}(x,y) = \left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 Z_{mn}(x,y) \tag{61}$$

$$\frac{\partial^4}{\partial y^4} Z_{mn}(x,y) = \frac{\partial^4}{\partial y^4} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{62}$$

$$\frac{\partial^4}{\partial y^4} Z_{mn}(x,y) = \left(\frac{n\pi}{b}\right)^4 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{63}$$

$$\frac{\partial^4}{\partial y^4} Z_{mn}(x,y) = \left(\frac{n\pi}{b}\right)^4 Z_{mn}(x,y) \tag{64}$$

$$+\left(\frac{m\pi}{a}\right)^4 D\left[\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{mn}(x,y)\, T_{mn}(t)\right]$$

$$+\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 2D\left[\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{mn}(x,y)\, T_{mn}(t)\right]$$

$$+\left(\frac{n\pi}{b}\right)^4 D\left[\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{mn}(x,y)\, T_{mn}(t)\right]$$

$$+\rho h\left[\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{mn}(x,y)\, \frac{\partial^2}{\partial t^2} T_{mn}(t)\right] = -\rho h \frac{\partial^2 w}{\partial t^2} \tag{65}$$

$$D\left[+\left(\frac{m\pi}{a}\right)^4+2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2+\left(\frac{n\pi}{b}\right)^4\right]\left[\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{mn}(x,y)\,T_{mn}(t)\right]$$

$$+\rho h\left[\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{mn}(x,y)\,\frac{\partial^2}{\partial t^2}T_{mn}(t)\right]=-\rho h\frac{\partial^2 w}{\partial t^2}$$

(66)

$$\frac{\rho h\,\omega^2}{D}=\left(\left(\frac{\pi m}{a}\right)^4+2\left(\frac{\pi m}{a}\right)^2\left(\frac{\pi n}{b}\right)^2+\left(\frac{\pi n}{b}\right)^4\right)$$

$$D\left[\frac{\rho h\,\omega^2}{D}\right]\left[\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{mn}(x,y)\,T_{mn}(t)\right]$$

$$+\rho h\left[\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{mn}(x,y)\,\frac{\partial^2}{\partial t^2}T_{mn}(t)\right]=-\rho h\frac{\partial^2 w}{\partial t^2}$$

(67)

$$\rho h\,\omega^2\left[\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{mn}(x,y)\,T_{mn}(t)\right]$$

$$+\rho h\left[\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{mn}(x,y)\,\frac{\partial^2}{\partial t^2}T_{mn}(t)\right]=-\rho h\frac{\partial^2 w}{\partial t^2}$$

(68)

Multiply each term by $Z_{uv}(x,y)$.

$$\rho h\,\omega^2\left[\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{uv}(x,y)\,Z_{mn}(x,y)\,T_{mn}(t)\right]$$

$$+\rho h\left[\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{uv}(x,y)\,Z_{mn}(x,y)\,\frac{\partial^2}{\partial t^2}T_{mn}(t)\right]=-\rho h\,Z_{uv}(x,y)\frac{\partial^2 w}{\partial t^2}$$

(69)

Integrate with respect to surface area.

$$\rho h\,\omega^2\int_0^b\int_0^a\left[\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{uv}(x,y)\,Z_{mn}(x,y)\,T_{mn}(t)\right]dxdy$$

$$+\rho h\int_0^b\int_0^a\left[\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{uv}(x,y)\,Z_{mn}(x,y)\,\frac{\partial^2}{\partial t^2}T_{mn}(t)\right]dxdy$$

$$=\ -\rho h\,\frac{\partial^2 w}{\partial t^2}\int_0^b\int_0^a\left[\sum_{m=1}^{q}\sum_{n=1}^{r} Z_{uv}(x,y)\right]dxdy$$

(70)

$$+\rho h\,\omega^2\left[\sum_{m=1}^{q}\sum_{n=1}^{r}\left[T_{mn}(t)\int_0^b\int_0^a Z_{uv}(x,y)\,Z_{mn}(x,y)\,dx\,dy\right]\right]$$

$$+\rho h\left[\sum_{m=1}^{q}\sum_{n=1}^{r}\left[\frac{\partial^2}{\partial t^2}T_{mn}(t)\int_0^b\int_0^a Z_{uv}(x,y)\,Z_{mn}(x,y)\,dx\,dy\right]\right]$$

$$= \quad -\rho h\,\frac{\partial^2 w}{\partial t^2}\left[\sum_{m=1}^{q}\sum_{n=1}^{r}\left[\int_0^b\int_0^a Z_{uv}(x,y)\right]\right] \tag{71}$$

The eigenvectors are orthogonal such that

$$\rho h\int_0^b\int_0^a Z_{uv}(x,y)\,Z_{mn}(x,y)\,dx\,dy = 0 \qquad \text{for} \quad u \neq m \quad \text{or} \quad v \neq n \tag{72}$$

$$\rho h\int_0^b\int_0^a Z_{uv}(x,y)\,Z_{mn}(x,y)\,dx\,dy = 1 \qquad \text{for} \quad u = m \quad \text{and} \quad v = n \tag{73}$$

$$\omega^2 \left[\sum_{m=1}^{q} \sum_{n=1}^{r} \left[T_{mn}(t) \right] \right] + \left[\sum_{m=1}^{q} \sum_{n=1}^{r} \left[\frac{\partial^2}{\partial t^2} T_{mn}(t) \right] \right]$$

$$= -\rho h \frac{\partial^2 w}{\partial t^2} \left[\sum_{m=1}^{q} \sum_{n=1}^{r} \left[\int_0^b \int_0^a Z_{mn}(x,y) dx\, dy \right] \right] \tag{74}$$

$$\frac{\partial^2}{\partial t^2} T_{mn}(t) + \omega^2 T_{mn}(t) = -\rho h \frac{\partial^2 w}{\partial t^2} \left[\int_0^b \int_0^a Z_{mn}(x,y) dx\, dy \right] \tag{75}$$

$$\frac{d^2}{dt^2} T_{mn}(t) + \omega^2 T_{mn}(t) = -\Gamma_{mn} \frac{d^2 w}{dt^2} \tag{76}$$

Add a damping term.

$$\frac{d^2}{dt^2} T_{mn}(t) + 2\xi_{mn} \omega_{mn}{}^2 T_{mn}(t) + \omega^2 T_{mn}(t) = -\Gamma_{mn} \frac{d^2 w}{dt^2} \tag{77}$$

$$\ddot{T}_{mn}(t) + 2\xi_{mn} \omega_{mn}{}^2 \dot{T}_{mn}(t) + \omega^2 T_{mn}(t) = -\Gamma_{mn} \ddot{w}(t) \tag{78}$$

The following solution is taken from Reference 3. The transfer function is

$$H(x,y,\omega) = \sum_{m=1}^{q} \sum_{n=1}^{r} \left\{ \frac{-\Gamma_{mn} Z_{mn}(x,y)}{(\omega_{mn}{}^{2} - \omega^{2}) + j2\xi_{mn}\,\omega\omega_{mn}} \right\} \tag{79}$$

$$H(x,y,\omega) = \frac{Z(x,y,\omega)}{\ddot{W}(\omega)} \tag{80}$$

The relative displacement is

$$Z(x,y,\omega) = \ddot{W}(\omega) \sum_{m=1}^{q} \sum_{n=1}^{r} \left\{ \frac{-\Gamma_{mn} Z_{mn}(x,y)}{(\omega_{mn}{}^{2} - \omega^{2}) + j2\xi_{mn}\,\omega\omega_{mn}} \right\} \tag{81}$$

$$Z(x,y,\omega) = \frac{2}{\sqrt{M}} \ddot{W}(\omega) \sum_{m=1}^{q} \sum_{n=1}^{r} \left\{ \frac{-\Gamma_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)}{(\omega_{mn}{}^{2} - \omega^{2}) + j2\xi_{mn}\,\omega\omega_{mn}} \right\} \tag{82}$$

The relative velocity is

$$\dot{Z}(x,y,\omega) = j\omega\,\ddot{W}(\omega) \sum_{m=1}^{q} \sum_{n=1}^{r} \left\{ \frac{-\Gamma_{mn} Z_{mn}(x,y)}{(\omega_{mn}{}^{2} - \omega^{2}) + j2\xi_{mn}\,\omega\omega_{mn}} \right\} \tag{83}$$

$$\dot{Z}(x,y,\omega) = j\omega \frac{2}{\sqrt{M}} \ddot{W}(\omega) \sum_{m=1}^{q} \sum_{n=1}^{r} \left\{ \frac{-\Gamma_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)}{(\omega_{mn}{}^{2} - \omega^{2}) + j2\xi_{mn}\,\omega\omega_{mn}} \right\} \tag{84}$$

The absolute acceleration is

$$\ddot{U}(x,y,\omega) = \ddot{W}(\omega)\left\{1+\sum_{m=1}^{q}\sum_{n=1}^{r}\left\{\frac{\omega^2\,\Gamma_{mn}\,Z_{mn}(x,y)}{(\omega_{mn}{}^2-\omega^2)+j2\xi_{mn}\,\omega\omega_{mn}}\right\}\right\} \quad (85)$$

$$\ddot{U}(x,y,\omega) = \ddot{W}(\omega)\left\{1+\frac{2\,\omega^2}{\sqrt{M}}\sum_{m=1}^{q}\sum_{n=1}^{r}\left\{\frac{\Gamma_{mn}\sin\left(\frac{m\pi\,x}{a}\right)\sin\left(\frac{n\pi\,y}{b}\right)}{(\omega_{mn}{}^2-\omega^2)+j2\xi_{mn}\,\omega\omega_{mn}}\right\}\right\} \quad (86)$$

The bending moments are

$$M_{xx}(x,y,\omega) = -D\left(\frac{\partial^2}{\partial x^2}+\mu\frac{\partial^2}{\partial y^2}\right)Z(x,y,\omega) \quad (87)$$

$$M_{yy}(x,y,\omega) = -D\left(\frac{\partial^2}{\partial y^2}+\mu\frac{\partial^2}{\partial x^2}\right)Z(x,y,\omega) \quad (88)$$

Let $\hat{z}$ be the distance from the centerline in the vertical axis.

The bending stresses from Reference 4 are

$$\sigma_{xx}(x,y,\omega) = -\frac{E\,\hat{z}}{1-\mu^2}\left(\frac{\partial^2}{\partial x^2}+\mu\frac{\partial^2}{\partial y^2}\right)Z(x,y,\omega) \quad (89)$$

$$\sigma_{xx}(x,y,\omega) = -\frac{E\hat{z}}{1-\mu^2}\ddot{W}(\omega)\sum_{m=1}^{q}\sum_{n=1}^{r}\left\{\frac{-\Gamma_{mn}\left(\frac{\partial^2}{\partial x^2}+\mu\frac{\partial^2}{\partial y^2}\right)Z_{mn}(x,y)}{(\omega_{mn}{}^2-\omega^2)+j2\xi_{mn}\,\omega\omega_{mn}}\right\} \tag{90}$$

$$\sigma_{xx}(x,y,\omega) = \frac{E\hat{z}}{1-\mu^2}\frac{2\pi^2}{\sqrt{M}}\ddot{W}(\omega)\sum_{m=1}^{q}\sum_{n=1}^{r}\left\{\frac{-\Gamma_{mn}\left(\frac{m^2}{a^2}+\mu\frac{n^2}{b^2}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)}{(\omega_{mn}{}^2-\omega^2)+j2\xi_{mn}\,\omega\omega_{mn}}\right\} \tag{91}$$

$$\sigma_{yy}(x,y,\omega) = -\frac{E\hat{z}}{1-\mu^2}\left(\frac{\partial^2}{\partial y^2}+\mu\frac{\partial^2}{\partial x^2}\right)Z(x,y,\omega) \tag{92}$$

$$\sigma_{yy}(x,y,\omega) = -\frac{E\hat{z}}{1-\mu^2}\ddot{W}(\omega)\sum_{m=1}^{q}\sum_{n=1}^{r}\left\{\frac{-\Gamma_{mn}\left(\frac{\partial^2}{\partial y^2}+\mu\frac{\partial^2}{\partial x^2}\right)Z_{mn}(x,y)}{(\omega_{mn}{}^2-\omega^2)+j2\xi_{mn}\,\omega\omega_{mn}}\right\} \tag{93}$$

$$\sigma_{yy}(x,y,\omega) = \frac{E\hat{z}}{1-\mu^2}\frac{2\pi^2}{\sqrt{M}}\ddot{W}(\omega)\sum_{m=1}^{q}\sum_{n=1}^{r}\left\{\frac{-\Gamma_{mn}\left(\frac{n^2}{b^2}+\mu\frac{m^2}{a^2}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)}{(\omega_{mn}{}^2-\omega^2)+j2\xi_{mn}\,\omega\omega_{mn}}\right\} \tag{94}$$

$$\tau_{xy}(x, y, \omega) = -\frac{E\,\hat{z}}{1+\mu}\left(\frac{\partial^2}{\partial x \partial y} Z(x, y, \omega)\right) \tag{95}$$

$$\tau_{xy}(x, y, \omega) = -\frac{E\,\hat{z}}{1+\mu}\ddot{W}(\omega)\sum_{m=1}^{q}\sum_{n=1}^{r}\left\{\frac{-\Gamma_{mn}\frac{\partial^2}{\partial x \partial y} Z_{mn}(x, y)}{(\omega_{mn}^{2} - \omega^2) + j2\xi_{mn}\,\omega\omega_{mn}}\right\} \tag{96}$$

$$\tau_{xy}(x, y, \omega) = -\frac{E\,\hat{z}}{1+\mu}\frac{2\pi^2}{\sqrt{M}}\ddot{W}(\omega)\sum_{m=1}^{q}\sum_{n=1}^{r}\left\{\frac{-\Gamma_{mn}\left(\frac{mn}{ab}\right)\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)}{(\omega_{mn}^{2} - \omega^2) + j2\xi_{mn}\,\omega\omega_{mn}}\right\} \tag{97}$$

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APPENDIX A

Normal Stress-Velocity Relationship

The maximum absolute spatial stresses for a given mode m,n are

$$\left|\sigma_{xx,mn}\right|_{max} = \frac{E\,\hat{z}}{1-\mu^2}\,\frac{2\pi^2}{\sqrt{M}}\,\ddot{W}(\omega)\left|\frac{\Gamma_{mn}\left(\frac{m^2}{a^2}+\mu\frac{n^2}{b^2}\right)}{(\omega_{mn}{}^2-\omega^2)+j2\xi_{mn}\,\omega\omega_{mn}}\right| \tag{A-1}$$

$$\left|\sigma_{yy,mn}\right|_{max} = \frac{E\,\hat{z}}{1-\mu^2}\,\frac{2\pi^2}{\sqrt{M}}\,\ddot{W}(\omega)\left|\frac{\Gamma_{mn}\left(\mu\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}{(\omega_{mn}{}^2-\omega^2)+j2\xi_{mn}\,\omega\omega_{mn}}\right| \tag{A-2}$$

The maximum spatial velocity for a given mode m,n is

$$\left|\dot{Z}_{mn}\right|_{max} = \frac{2\omega}{\sqrt{M}}\,\ddot{W}(\omega)\left|\frac{\Gamma_{mn}}{(\omega_{mn}{}^2-\omega^2)+j2\xi_{mn}\,\omega\omega_{mn}}\right| \tag{A-3}$$

Thus

$$\left|\sigma_{xx,mn}\right|_{max} = \frac{E\,\hat{z}}{1-\mu^2}\,\frac{\pi^2}{\omega}\left(\frac{m^2}{a^2}+\mu\frac{n^2}{b^2}\right)\left|\dot{Z}_{mn}\right|_{max} \tag{A-4}$$

$$\left|\sigma_{yy,mn}\right|_{max} = \frac{E\,\hat{z}}{1-\mu^2}\,\frac{\pi^2}{\omega}\left(\mu\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\left|\dot{Z}_{mn}\right|_{max} \tag{A-5}$$