THE NATURAL FREQUENCY OF A RECTANGULAR
PLATE POINT-SUPPORTED AT EACH CORNER
Revision C

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August 1, 2011

Introduction

The Rayleigh method is used in this tutorial to determine the fundamental bending frequency. A displacement function is assumed which satisfies the geometric boundary conditions. The geometric conditions are the displacement and slope conditions at the boundaries.

The assumed displacement function is substituted into the strain and kinetic energy equations.

The Rayleigh method gives a natural frequency that is an upper limited of the true natural frequency. The method would give the exact natural frequency if the true displacement function were used. The true displacement function is called an eigenfunction.

Consider the rectangular plate in Figure 1.

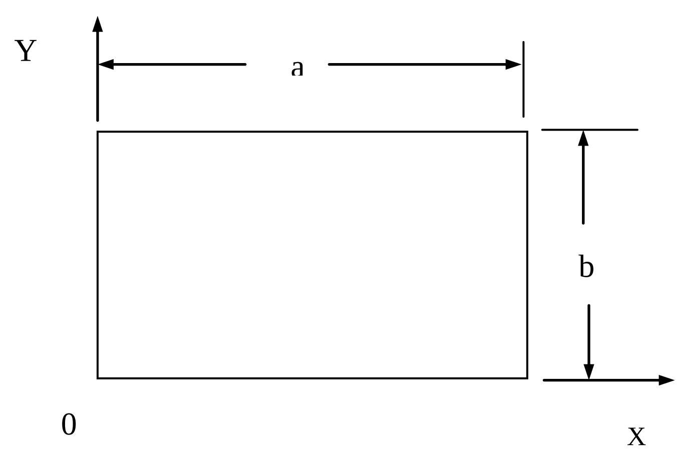

Figure 1.

Let Z represent the out-of-plane displacement. The total strain energy V of the plate is

$$V = \frac{D}{2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\left[\left(\frac{\partial^2 Z}{\partial X^2}\right)^2 + \left(\frac{\partial^2 Z}{\partial Y^2}\right)^2 + 2\mu\left(\frac{\partial^2 Z}{\partial X^2}\right)\left(\frac{\partial^2 Z}{\partial Y^2}\right) + 2(1-\mu)\left(\frac{\partial^2 Z}{\partial X \partial Y}\right)^2\right] dX dY \tag{1}$$

Note that the plate stiffness factor D is given by

$$D = \frac{Eh^3}{12\ (1-\mu^2\ )} \tag{2}$$

where

E = elastic modulus
h = plate thickness
μ = Poisson's ratio

The total kinetic energy T of the plate bending is given by

$$T = \frac{\rho h \Omega^2}{2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} Z^2\ dX dY \tag{3}$$

where

ρ = mass per volume
Ω = angular natural frequency

Plate Fixed at Each Corner

Consider the plate in Figure 2. The plate is point-supported at each corner, but the edges are free in between the corners. The point supports prevent displacement but allow rotation.

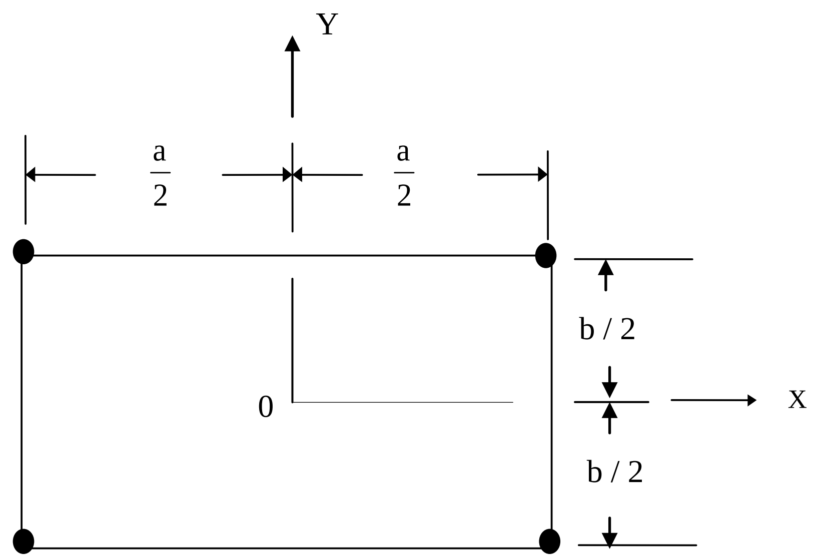

Figure 2.

Seek a displacement function Z(x, y). The geometric boundary conditions are

$$Z(x, y) = 0 \quad \text{for x and y at each corner.} \tag{4}$$

$$Z(0,0) = Z_o \tag{5}$$

$$Z(x,0) = \alpha Z_o \quad \text{for} \quad x = \pm\, a/2 \quad \text{and} \quad 0 \le \alpha \le 1 \tag{6}$$

$$Z(0, y) = \beta Z_o \quad \text{for} \quad y = \pm\, b/2 \quad \text{and} \quad 0 \le \beta \le 1 \tag{7}$$

Furthermore, bending moment condition is

$$M_x = -D\left[\frac{\partial^2 Z}{\partial x^2} + \mu \frac{\partial^2 Z}{\partial y^2}\right] = 0 \qquad \text{for } \ x \text{ and } y \text{ at each corner.} \tag{8}$$

$$M_y = -D\left[\frac{\partial^2 Z}{\partial y^2} + \mu \frac{\partial^2 Z}{\partial x^2}\right] = 0 \qquad \text{for } \ x \text{ and } y \text{ at each corner.} \tag{9}$$

Let

$$Z_o = 1 \tag{10}$$

The candidate displacement function is

$$Z(x, y) = c_1 \cos\left(\frac{\pi x}{a}\right) + c_2 \cos\left(\frac{\pi y}{b}\right) + c_3 \cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right) \tag{11}$$

The candidate displacement function satisfies the geometric boundary conditions. It also satisfies the bending moment boundary conditions. It does not satisfy the twisting moment and shear boundary conditions, however.

Now equate the total kinetic energy with the total strain energy per Rayleigh's method, using equations 1-3 and 11. This is done numerically via the computer program in Appendix A. The integrals are converted to series form for this calculation.

Solve for Ω. Select c_1, c_2 and c_3 values to minimize Ω via trial-and-error. The c_3 value is constrained to be less than the other two coefficients.

The natural frequency f_n is

$$f_n \approx \frac{1}{2\pi}\Omega \tag{12}$$

A more proper equation is

$$f_n \leq \frac{1}{2\pi}\Omega \tag{13}$$

Verification

The following formula taken from Steinberg's text can be used to determine the frequency bounds.

$$\frac{1.13}{a^2}\sqrt{\frac{D}{\rho}} \le f_n \le \frac{1.13}{b^2}\sqrt{\frac{D}{\rho}} \quad , \qquad \text{for } a > b \tag{14}$$

An additional verification method is given in Appendix A.

Example

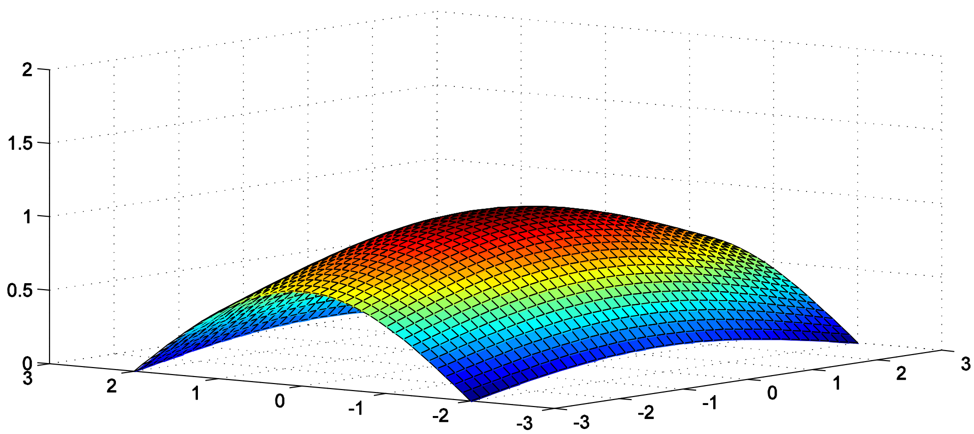

Figure 3.

An aluminum plate has dimensions of 6 x 4 x 0.063 inches. It is point-supported at each corner. The elastic modulus is 1.0e+07 lbf/in^2. The mass density is 0.1 lbm/in^3.

The fundamental frequency is 151.6 Hz, as calculated using the trial-and-error Rayleigh method outlined above. The resulting mode shape is shown in Figure 3.

The modal displacement equation is

$$Z(x,y) = 0.6438\cos\left(\frac{\pi x}{6}\right) + 0.1791\cos\left(\frac{\pi y}{4}\right) + 0.1771\cos\left(\frac{\pi x}{6}\right)\cos\left(\frac{\pi y}{4}\right) \tag{15}$$

Again, the fundamental frequency is 151.6 Hz.

The expected natural frequency range per equation (14) is

$$117.6 < f_n < 264.5 \text{ Hz}$$

The frequency calculated using the handbook formula in Appendix A is 147.7 Hz.

The Rayleigh method accuracy can be improved using the Rayleigh-Ritz method.

References

1. R. Blevins, Formulas for Natural Frequency and Mode Shape, Krieger, Malabar, Florida, 1979. See Table 11-6.
2. D. Steinberg, Vibration Analysis for Electronic Equipment, Third Edition, Wiley, New York, 2000.

APPENDIX A

The natural frequency for a point supported rectangular plate from Blevin's text is

$$f_1 = \frac{\lambda_1^2}{2\pi a^2}\sqrt{\frac{D}{\rho}} \qquad \text{(A-1)}$$

where

a/b	λ_1^2
1.0	7.12
1.5	8.92
2.0	9.29
2.5	9.39

The plate stiffness factor is

$$D = \frac{Eh^3}{12\left(1-\mu^2\right)} \qquad \text{(A-2)}$$

where

E = elastic modulus
h = plate thickness
μ = Poisson's ratio
ρ = mass/area