INTEGRATION OF THE POWER SPECTRAL DENSITY FUNCTION Revision B

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Introduction

Random vibration is represented in the frequency domain by a power spectral density function. The overall root-mean-square (RMS) value is equal to the square root of the area under the curve. The purpose of this tutorial is to explain the integration procedure.

A power spectral density specification is typically represented as follows:

1. The specification is represented as a series of piecewise continuous segments.
2. Each segment is a straight line on a log-log plot.

An example is shown in Figure 1.

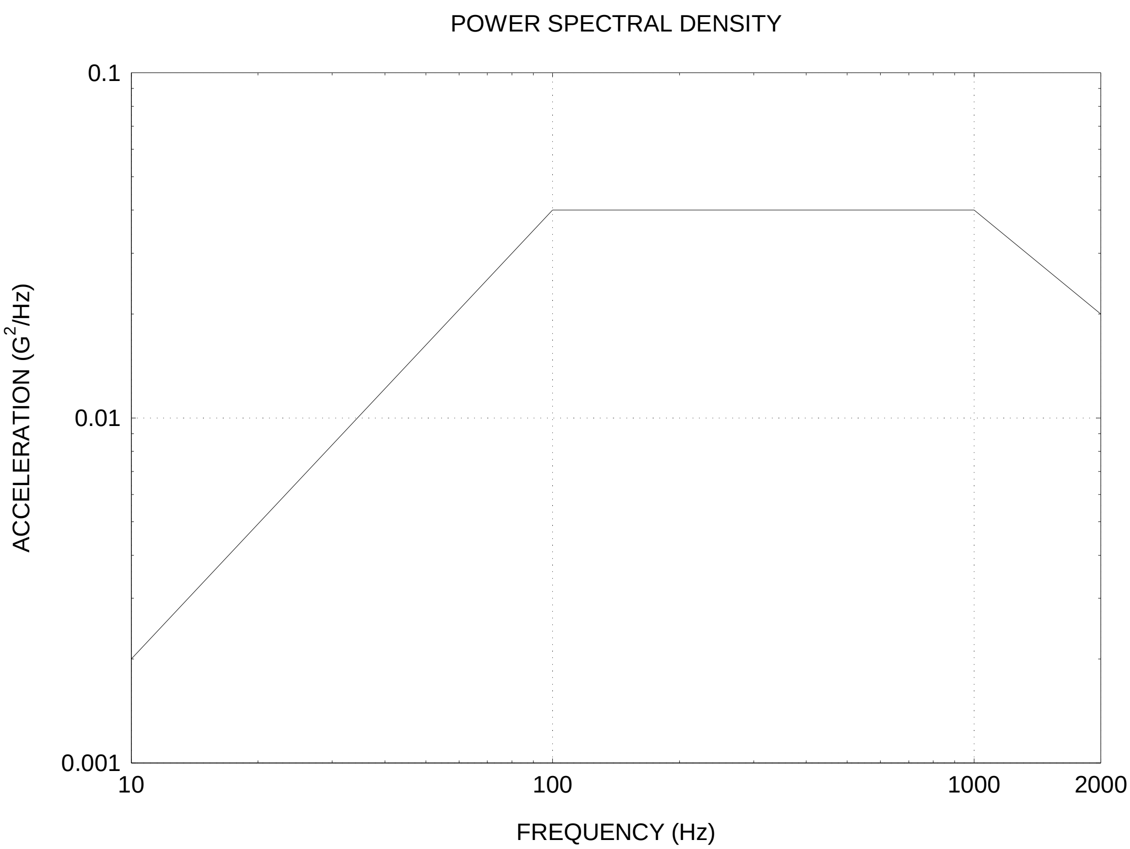

Figure 1.

Note that the power spectral density amplitude is represented in units of (G^2/Hz). This is an abbreviated notation. The actual unit is (G_{RMS}^2/Hz).

Derivation

The equation for each segment is

$$y(f) = \left[\frac{y_1}{f_1^{\,n}}\right] f^{\,n} \tag{1}$$

The starting coordinate is (f_1, y_1).

The exponent n is a real number which represents the slope. The slope between two coordinates (f_1, y_1) and (f_2, y_2) is

$$n = \frac{\log\left(\frac{y_2}{y_1}\right)}{\log\left(\frac{f_2}{f_1}\right)} \tag{2}$$

The area a_1 under segment 1 is

$$a_1 = \int_{f_1}^{f_2} \left[\frac{y_1}{f_1^{\,n}}\right] f^{\,n} df \tag{3}$$

There are two cases depending on the exponent n.

The first case is

$$a_1 = \left[\frac{y_1}{f_1^{\,n}}\right]\left[\frac{1}{n+1}\right] f^{\,n+1}\Big|_{f_1}^{f_2} \quad , \quad \text{for } n \neq -1 \tag{4}$$

$$a_1 = \left[\frac{y_1}{f_1^{\,n}}\right]\left[\frac{1}{n+1}\right]\left[f_2^{\,n+1} - f_1^{\,n+1}\right], \quad \text{for } n \neq -1 \tag{5}$$

The second case is

$$a_1 = \int_{f_1}^{f_2} \left[\frac{y_1}{f_1^{-1}} \right] f^{-1}\, df \,, \quad \text{for } n = -1 \tag{6}$$

$$a_1 = \int_{f_1}^{f_2} [y_1 f_1] \frac{df}{f} \,, \quad \text{for } n = -1 \tag{7}$$

$$a_1 = [y_1 f_1] \ln(f) \Big|_{f_1}^{f_2} , \quad \text{for } n = -1 \tag{8}$$

$$a_1 = [y_1 f_1] \left[\ln(f_2) - \ln(f_1) \right], \quad \text{for } n = -1 \tag{9}$$

$$a_1 = [y_1 f_1] \left[\ln\left(\frac{f_2}{f_1} \right) \right], \quad \text{for } n = -1 \tag{10}$$

In summary, the area under segment i is

$$a_i = \begin{cases} \left[\dfrac{y_i}{f_i^{\,n}} \right] \left[\dfrac{1}{n+1} \right] \left[f_{i+1}^{\,n+1} - f_i^{\,n+1} \right], & \text{for } n \neq -1 \\ [y_i\, f_i] \left[\ln\left(\dfrac{f_{i+1}}{f_i} \right) \right], & \text{for } n = -1 \end{cases} \tag{11}$$

The overall level L is

$$L = \sqrt{\sum_{i=1}^{m} a_i} \tag{12}$$

where m is the total number of segments.

Example

Consider the power spectral density function in Figure 1. The breakpoints are given in Table 1.

Table 1. Power Spectral Density	
Freq (Hz)	Level (G^2/Hz)
10	0.002
100	0.04
1000	0.04
2000	0.02

Consider the first pair of coordinates:

f_1 = 10 Hz	y_1 = 0.002 G^2/Hz
f_2 = 100 Hz	y_2 = 0.04 G^2/Hz

Calculate the slope.

$$n = \frac{\log\left(\frac{0.04}{0.002}\right)}{\log\left(\frac{100}{10}\right)} \tag{13}$$

$$n = 1.3 \tag{14}$$

Substitute into equation (11).

$$a_1 = \left[\frac{0.002}{10^{1.3}}\right]\left[\frac{1}{1.3+1}\right]\left[100^{1.3+1} - 10^{1.3+1}\right] \tag{15}$$

$$a_1 = \left[\frac{0.002}{10^{1.3}}\right]\left[\frac{1}{2.3}\right]\left[100^{2.3} - 10^{2.3}\right] \tag{16}$$

$$a_1 = 1.726 \ \ G^2 \tag{17}$$

Consider the second pair:

f_2 = 100 Hz	y_2 = 0.04 G^2/Hz
f_3 = 1000 Hz	y_3 = 0.04 G^2/Hz

Calculate the slope.

$$n = \frac{\log\left(\frac{0.04}{0.04}\right)}{\log\left(\frac{1000}{100}\right)} \tag{18}$$

$$n = 0. \tag{19}$$

Substitute into equation (11).

$$a_2 = \left[\frac{0.04}{100^0}\right]\left[\frac{1}{0+1}\right]\left[1000^{0+1} - 100^{0+1}\right] \tag{20}$$

$$a_2 = \left[\frac{0.04}{1}\right]\left[\frac{1}{1}\right]\left[1000^1 - 100^1\right] \tag{21}$$

$$a_2 = 36.000 \ \ G^2 \tag{22}$$

Consider the third pair:

f_3 = 1000 Hz	y_3 = 0.04 G^2/Hz
f_4 = 2000 Hz	y_4 = 0.02 G^2/Hz

Calculate the slope.

$$n = \frac{\log\left(\frac{0.02}{0.04}\right)}{\log\left(\frac{2000}{1000}\right)} \tag{23}$$

$$n = -1. \tag{24}$$

Substitute into equation (11).

$$a_3 = [(0.04)(1000)]\left[\ln\left(\frac{2000}{1000}\right)\right] \quad (25)$$

$$a_3 = 27.726 \quad (26)$$

Now substitute the individual area values into equation (12).

$$L = \sqrt{(1.726 + 36.000 + 27.726)G^2} \quad (27)$$

The overall level is

$$L = 8.09\ G_{RMS} \quad (28)$$

Additional information on slopes is given in Appendix A.

APPENDIX A

Introduction to dB/octave Slopes

NAVMAT P-9492 gives the power spectral density specification shown in Figure A-1.

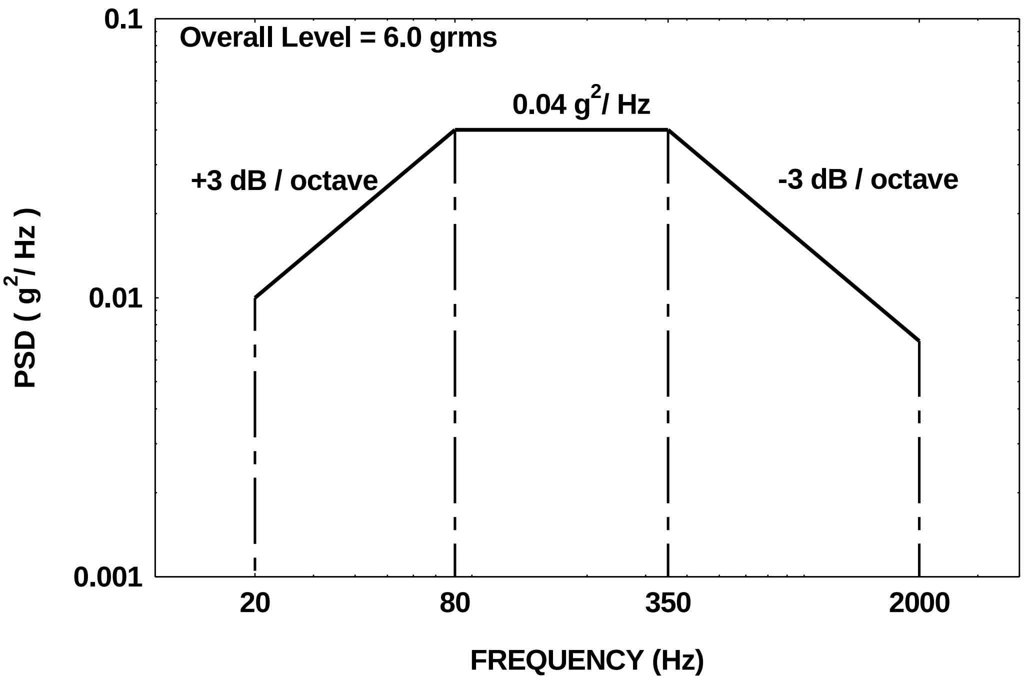

Figure A-1.

The task is to determine the coordinates of the endpoints.

Derivation

Assume that a_1 and a_2 each has an amplitude in G^2/Hz. The difference in dB between a_1 and a_2 is

$$\Delta dB = 10 \log \left[\frac{a_2}{a_1} \right] \qquad \text{(A-1)}$$

Furthermore,

$$a_2 = a_1 \left[10^{\Delta dB\ /10} \right] \tag{A-2}$$

Additional equations are needed.

The slope N between two coordinates (f_1, a_1) and (f_2, a_2) in a log-log plot is

$$N = \frac{\log\left[\frac{a_2}{a_1}\right]}{\log\left[\frac{f_2}{f_1}\right]} \tag{A-3}$$

Solve for a_2 .

$$N \log\left[\frac{f_2}{f_1}\right] = \log\left[\frac{a_2}{a_1}\right] \tag{A-4}$$

$$\log\left\{\left[\frac{f_2}{f_1}\right]^N\right\} = \log\left[\frac{a_2}{a_1}\right] \tag{A-5}$$

Take the anti-log.

$$\left[\frac{f_2}{f_1}\right]^N = \left[\frac{a_2}{a_1}\right] \tag{A-6}$$

$$\left[\frac{a_2}{a_1}\right] = \left[\frac{f_2}{f_1}\right]^N \tag{A-7}$$

Thus,

$$a_2 = a_1\left[\frac{f_2}{f_1}\right]^N \tag{A-8}$$

Now consider a one-octave frequency separation.

$$f_2 = 2 f_1 \tag{A-9}$$

Substitute equation (A-9) into (A-3).

$$N = \frac{\log\left[\frac{a_2}{a_1}\right]}{\log[2]} \tag{A-10}$$

Substitute equation (A-1) into (A-10).

$$N = \frac{\Delta dB / 10}{\log[2]} \tag{A-11}$$

Note that ΔdB represents the dB/octave slope in equation (A-11). Again, equations (A-10) and (A-11) assume a one-octave frequency separation.

Now substitute equation (A-11) into (A-8).

$$a_2 = a_1 \left[\frac{f_2}{f_1}\right]^{\left\{\frac{\Delta dB / 10}{\log[2]}\right\}} \tag{A-12}$$

Example

Calculate the amplitude at 2000 Hz for the power spectral density in Figure A-1. The slope is -3 dB/octave.

Note

$f_1 = 350$ Hz

$f_2 = 2000$ Hz

$a_1 = 0.04\ G^2 / Hz$

Substitute into equation (A-12).

$$a_2 = 0.04\ G^2/\mathrm{Hz}\left[\frac{2000\ \mathrm{Hz}}{350\ \mathrm{Hz}}\right]^{\left\{\frac{-3\ \mathrm{dB}/10}{\log[2]}\right\}} \tag{A-13}$$

$$a_2 = 0.007\ G^2/\mathrm{Hz} \quad \text{at 2000 Hz} \tag{A-14}$$

Now calculate the amplitude at 20 Hz for the power spectral density in Figure A-1. The slope is +3dB/octave.

Note

$f_1 = 80$ Hz

$f_2 = 20$ Hz

$a_1 = 0.04\ G^2/\mathrm{Hz}$

Substitute into equation (A-12). Note that this equation allows $f_2 < f_1$.

$$a_2 = 0.04\ G^2/\mathrm{Hz}\left[\frac{20\ \mathrm{Hz}}{80\ \mathrm{Hz}}\right]^{\left\{\frac{+3\ \mathrm{dB}/10}{\log[2]}\right\}} \tag{A-15}$$

$$a_2 = 0.01\ G^2/\mathrm{Hz} \quad \text{at 20 Hz} \tag{A-16}$$