ROCKET PROPULSION EQUATION

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Consider the powered flight of a rocket vehicle. Assume the following:

1. The rocket may have an initial velocity (as would be the case for an upper stage burn).
2. The rocket is initially unpowered.
3. The atmospheric drag force is negligible.
4. Gravity is negligible.
5. The propellant mass flow rate is constant.
6. The exhaust velocity relative to the rocket is constant.
7. The pressure at the nozzle exit plane is equal to the ambient pressure.

A diagram of the rocket is shown for two time cases in Figure 1.

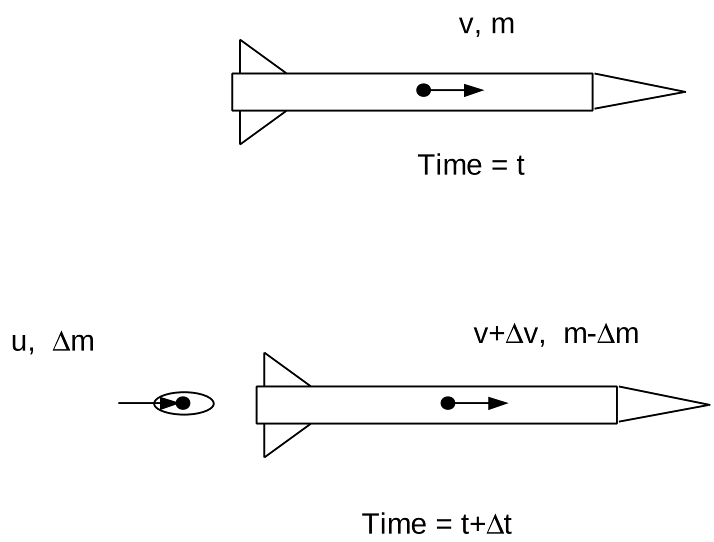

Figure 1.

Note that

v = absolute rocket velocity
u = absolute exhaust gas velocity
m = mass

The initial momentum of the system P_i is

$$P_i = mv \tag{1}$$

The final momentum of the system P_f is

$$P_f = (m - \Delta m)(v + \Delta v) + \Delta m\, u \tag{2}$$

The change in momentum ΔP is

$$\Delta P = P_f - P_i \tag{3}$$

$$\Delta P = [(m - \Delta m)(v + \Delta v) + \Delta m\, u] - mv \tag{4}$$

$$\Delta P = mv + m\Delta v - \Delta m v - \Delta m \Delta v + \Delta m u - mv \tag{5}$$

$$\Delta P = m\Delta v - v\Delta m - \Delta m \Delta v + \Delta m u \tag{6}$$

$$\Delta P = [u - (v + \Delta v)]\Delta m + m\,\Delta v \tag{7}$$

Let c be the exhaust gas velocity relative to the rocket. Recall that c is assumed to be constant.

$$c = [u - (v + \Delta v)] \tag{8}$$

Substitute equation (8) into (7).

$$\Delta P = c\,\Delta m + m\Delta v \tag{9}$$

The change in momentum with respect to time is

$$\frac{\Delta P}{\Delta t} = c\,\frac{\Delta m}{\Delta t} + m\,\frac{\Delta v}{\Delta t} \tag{10}$$

Take the limit as the time interval approaches zero.

$$\frac{dP}{dt} = c\,\frac{dm}{dt} + m\,\frac{dv}{dt} \tag{11}$$

Now let F be an external force applied to the system. Newton's second law is

$$F = \frac{dP}{dt} \tag{12}$$

The external force applied to the rocket is zero, however. The thrust is generated internally. Thus

$$\frac{dP}{dt} = 0 \tag{13}$$

Substitute equation (11) into (13).

$$c\,\frac{dm}{dt} + m\,\frac{dv}{dt} = 0 \tag{14}$$

$$c\,\frac{dm}{dt} = -m\,\frac{dv}{dt} \tag{15}$$

Equation (15) governs the flight of the rocket per the previously given assumptions.

Now solve for the final velocity. Multiply through by dt.

$$c\,dm = -m\,dv \tag{16}$$

$$-c\,\frac{dm}{m} = dv \tag{17a}$$

$$dv = -c\,\frac{dm}{m} \tag{17b}$$

Let

m_o = initial rocket mass

m_f = final rocket mass

v_o = initial rocket velocity

v_f = final rocket velocity

Integrate equation (17b).

$$\int_{v_o}^{v_f} dv = -\int_{m_o}^{m_f} c \frac{dm}{m} \tag{18}$$

$$v_f - v_o = -c \ln(m)\Big|_{m_o}^{m_f} \tag{19}$$

The final velocity is

$$v_f = c \ln\left(\frac{m_o}{m_f}\right) + v_o \tag{20}$$

Reference

1. Halliday and Resnik, Physics Parts I & II, Wiley, New York, 1978.