Lateral Natural Frequency of a Shaft Rotor System by the Transfer Matrix Method,
Revision A

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Abstract

Three analytical methods namely, Dunkerley's method, the Rayleigh Method and the Transfer Matrix method, are discussed for evaluating lateral natural frequencies of a shaft rotor system. One hypothetical case is considered and the natural frequency is evaluated by these methods. A program is developed by the Transfer Matrix Method. It may be useful for evaluating the natural frequency of a fiberizer for the sugar industry or a crusher for the cement industry.

Introduction

A critical speed of a rotating shaft is the speed at which the shaft starts to vibrate violently in the transverse direction. It is very dangerous to continue to run the shaft at its critical speed because the amplitude of vibration will build to such a level that the system may break into pieces. The phenomenon of bending vibrations and critical speeds of rotating shafts is perhaps the most common problem that is discussed by the vibration engineer, as it is a vexing day to day problem in design and maintenance of the machinery. Some of the rotors weigh as much as 100 tons as is the case of large steam turbines, and obviously they deserved the utmost attention in this regard.

The rotors always have some amount of residual unbalance however well they are balanced, and will experience resonance when they rotate at speeds equal to the bending natural frequency. These speeds are called critical speeds, and as far as possible they should be avoided. Even while taking the rotor through a critical speed to an operation speed, special precautions should be taken.

While calculation of the bending natural frequency of a simple shaft in rigid bearings is somewhat an easy matter, the problem in practice becomes complex because of:

1. Gyroscopic effects of disks
2. Dissimilar moment of area of shaft
3. Stiffness and damping properties of oil film bearings
4. Coupling between two rotors

To avoid failures of shafting, the general practice in the design of rotors is to determine the bending critical speeds, check the out-of-balance response and adopt a suitable balancing procedure. We know that a single mass flexible rotor with residual unbalance has critical shaft speeds which are the same as the natural frequencies of the rotor in lateral bending. In practice, the rotor carries several components, such as gears, disks, flywheels, etc. Such a rotor has several critical speeds corresponding to the bending natural frequencies. For most of the rotors, it is the fundamental mode which fails in the running speed zone. There are several methods of calculation of the critical speed, which are as follows:

1. Dunkerley method
2. Rayleigh method
3. Transfer Matrix method

The first two methods are suitable for estimating the fundamental frequency by hand calculation. The transfer matrix technique inevitably necessitates the use of computers.

Dunkerley Method

This is a very convenient method, proposed by Dunkerley to determine the fundamental critical speed of a shaft carrying a number of components. This method is quite simple and consists of reducing the actual system into a number of simple subsystems, calculating the critical speeds of each by a direct formula, and combing these critical speeds according to equation 1 to obtain the actual critical speed ω of the system.

$$\frac{1}{\omega^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \cdots + \frac{1}{\omega_n^2} \tag{1}$$

where $\omega_1, \omega_2, \omega_3, \ldots$ are the natural frequencies of the different rotor subsystems.

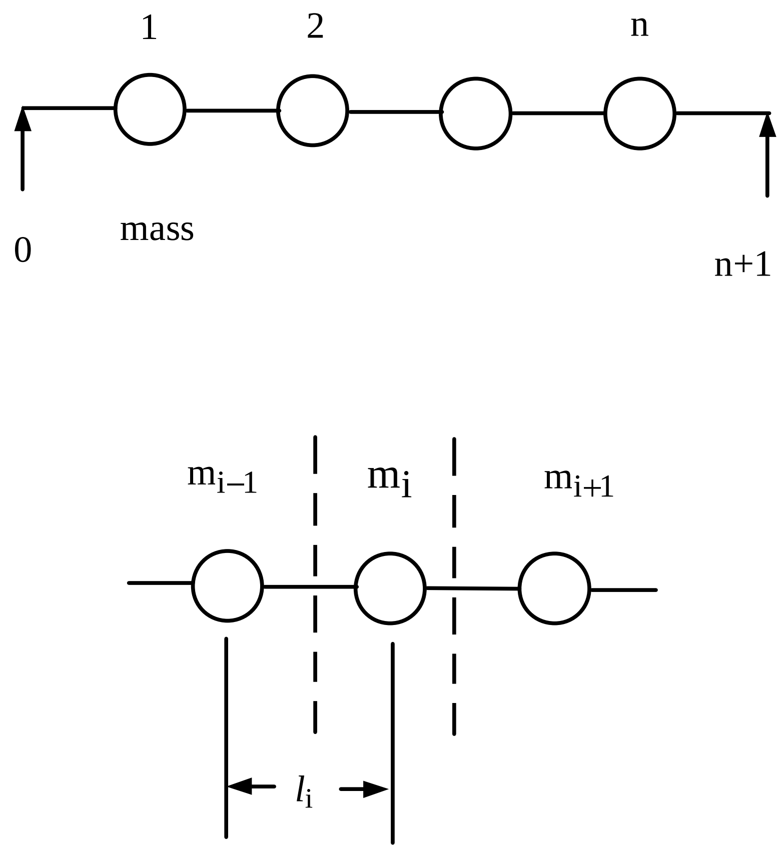


Figure 1. A *n* mass system

Rayleigh Method

This is another simple method, proposed by Rayleigh, based on the fact that the maximum kinetic energy must be equal to the maximum potential energy for a conservative system under a free vibration condition. For a shaft carrying several components, we can use static deflection or any other suitable function to represent the fundamental mode of the shaft. The fundamental frequency can be obtained from Reference [2].

$$\omega^2 = \frac{g \sum M_i y_i}{\sum M_i {y_i}^2} \tag{2}$$

where M_1, M_2, M_3, … are masses of different rotors and y_1, y_2, y_3, … are deflections of the shaft at the locations of these components. The variable g is the gravitational constant. Since the frequency is a minimum, it is always an upper bound.

Transfer Matrix Method

In a manner similar to the Holzer method, Myklestad and Prohl [3] developed a highly successful method of computation for the bending critical speeds of the shaft. Consider an *n* mass system, each mass representing either a gear, a disk, or a flywheel, etc. All of the masses are taken as lumped with their gyroscopic inertia neglected. The i^{th} shaft of length l_i and mass m_i are shown separately in Figure 1; and [S] represents the state vector containing the deflection y, slope θ, bending moment M, and shear force V. The sign convention of the variables is shown in Figure 2. The transfer matrices for each element are set up as follows:

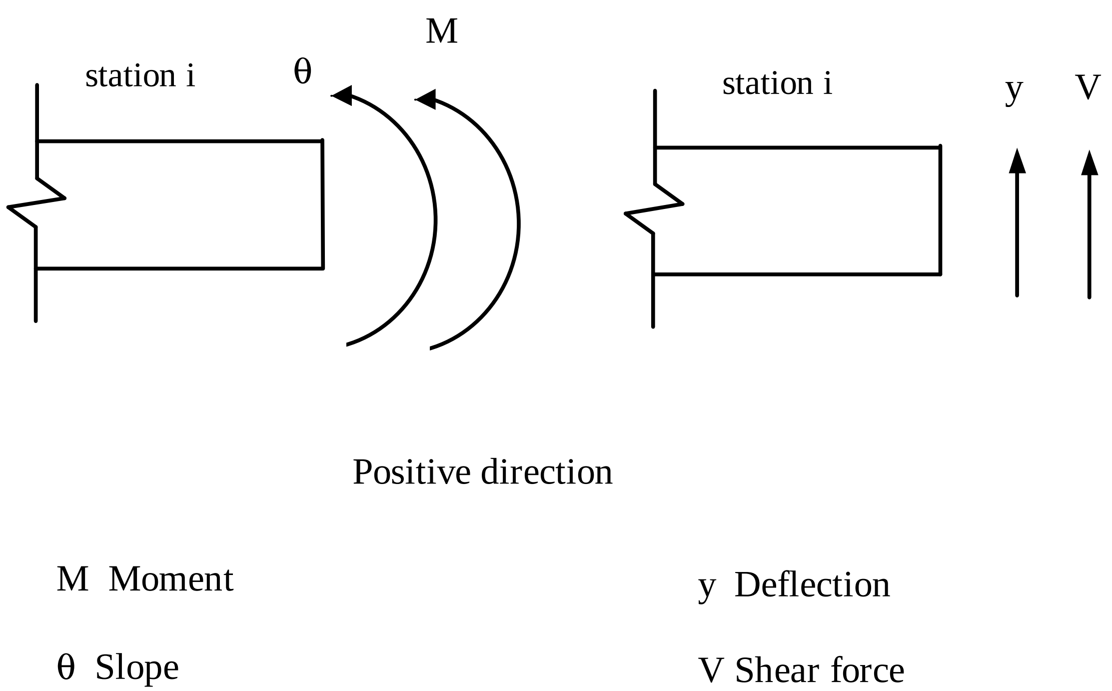


Figure 2. Sign convention for variables

Field Matrix

Figure 3 gives the equilibrium relation for the i^{th} field, from which we have in terms of shear force and bending moment,

$$V_i^L = V_{i-1}^R \tag{3}$$

$$M_i^L = M_{i-1}^R + V_{i-1}^R l_{i-1} \tag{4}$$

To derive the transfer relation for definition y and slope θ, for the element in Figure 3, we use the relation for a cantilever beam from Reference [4].

$$y = \frac{ML^2}{2EI} - \frac{Vl^3}{3EI} \tag{5}$$

$$\theta = \frac{Ml}{EI} - \frac{Vl^2}{2EI} \tag{6}$$

Using the above equations, we can obtain the following relations for a deflection y, slope θ to the left of station *i*, in terms of corresponding quantities to the right of station *i*-1.

$$y_i^L = y_{i-1}^R + \theta_{i-1}^R l_i + \frac{M_i^L l_i^2}{2EI_i} - \frac{V_i^L l_i^3}{3EI_i} \tag{7}$$

$$\theta_i^L = \theta_{i-1}^R + \frac{M_i^L l_i}{EI_i} - \frac{V_i^L l_i^2}{2EI_i} \tag{8}$$

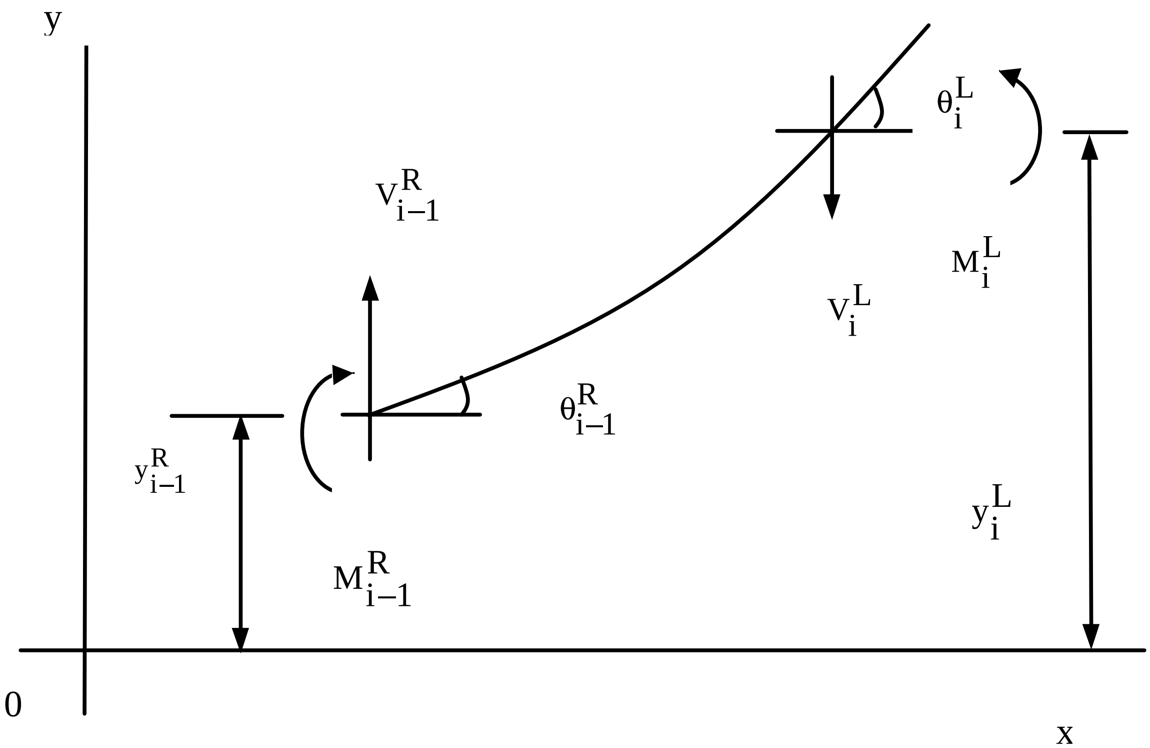
Using equations 3 and 4 in equations 7 and 8, and simplifying, we obtain

$$y_i^L = y_{i-1}^R + \theta_{i-1}^R l_i + \frac{M_{i-1}^R l_i^2}{2EI_i} + \frac{V_{i-1}^R l_i^3}{6EI_i} \tag{9}$$

$$\theta_i^L = \theta_{i-1}^R + \frac{M_{i-1}^R l_i}{EI_i} - \frac{V_{i-1}^R l_i^2}{2EI_i} \tag{10}$$

We combine equations 3, 4, and 7, and 8 in the following transfer matrix.

$$\begin{bmatrix} y \\ \theta \\ M \\ V \end{bmatrix}_i^L = \begin{bmatrix} 1 & l & \frac{l^2}{2EI} & \frac{l^3}{6EI} \\ 0 & 1 & \frac{l}{EI} & \frac{l^2}{2EI} \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix}_i \begin{bmatrix} y \\ \theta \\ M \\ V \end{bmatrix}_{i-1}^R \tag{11}$$



Elastic deformation of shaft (field) section

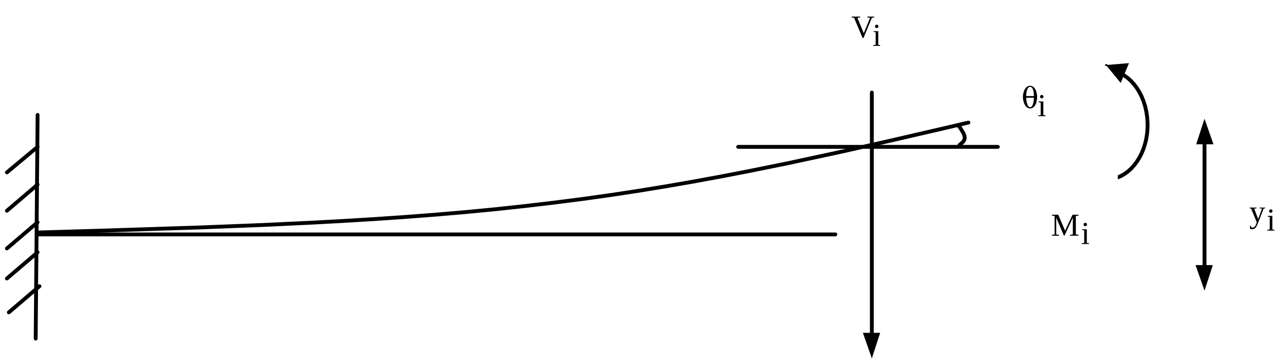


Figure 3. Relation for the i^{th} field

Symbolically, the above equation can be written as

$$[S]_i^L = [F]_i [S]_{i-1}^R \tag{12}$$

where $[F]_i$ is a field matrix of the i^{th} field.

Point Matrix

Figure 4 shows the equilibrium relations for the mass at station *i*. We can directly write the following transfer matrix

$$\begin{bmatrix} y \\ \theta \\ M \\ V \end{bmatrix}_i^R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m\omega^2 & 0 & 0 & 1 \end{bmatrix}_i \begin{bmatrix} y \\ \theta \\ M \\ V \end{bmatrix}_i^L \tag{13}$$

i.e.

$$[S]_i^R = [P]_i [S]_i^L \tag{14}$$

In the above equation, $[P]_i$ is the point matrix for the i^{th} mass having vibration in a normal mode with frequency ω.

Overall Transfer Matrix and Frequency Equation

Starting from station 0 of the shaft in Figure 1, we can write the following equations using equations 12 and 14.

$$[S]_1^L = [F]_1 [S]_0^R \tag{15}$$

$$[S]_1^R = [P]_1 [S]_1^L = [P]_1 [F]_1 [S]_0^R \tag{16}$$

$$[S]_2^L = [F]_2 [S]_1^R = [F]_2 \, [P]_1 [F]_1 [S]_0^R \tag{17}$$

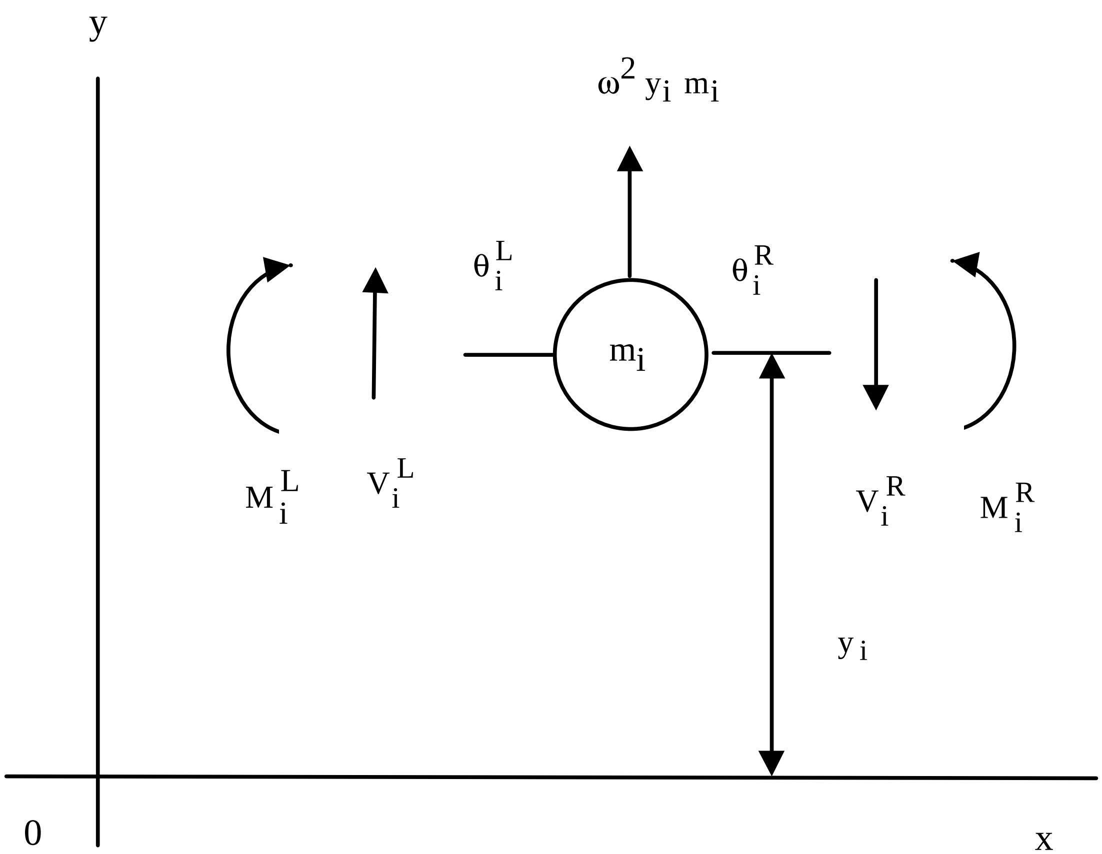


Figure 4. Relation for i^{th} mass

A generalized equation can be written as

$$[S]_{n+1}^{L} = [F]_{n+1}[P]_n[F]_n[P]_{n-1}\ldots[F]_1[S]_0^R \tag{18}$$

Defining the product of all field matrices and point matrices in the above equations as the overall transfer matrix [U], we obtain

$$[S]_{n+1} = [U]\,[S]_0^R \tag{19}$$

Since [F] and [P] are all 4 x 4 matrices, [U] is also 4 x 4 in size. We can write equation 19 in the expanded form

$$\begin{bmatrix} y \\ \theta \\ M \\ V \end{bmatrix}_i^R = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{bmatrix}_i \begin{bmatrix} y \\ \theta \\ M \\ V \end{bmatrix}_0 \qquad (20)$$

For a simply supported shaft, the bending moment M and the deflection y have boundary conditions

$$y = 0; \qquad M=0; \quad \text{at station 0 and } (n+1) \qquad (21)$$

Since θ and V are non-zero at station 0 and n+1, we obtain a determinant from equation 21

$$\Delta = \begin{vmatrix} u_{12} & u_{14} \\ u_{32} & u_{34} \end{vmatrix} = 0 \qquad (22)$$

We adopt a root searching technique to determine the natural frequency ω. Find the value of ω^2, to satisfy equation 22. Three examples are shown.

Example: By Rayleigh Method

A uniform shaft with two disks and supported bearings is shown in Figure 5. The calculated static shaft deflection is indicated in Figure 5b. Evaluate the natural frequency of the system.

Solution: We have the Rayleigh principle in which

$$\omega^2 = \frac{g \sum M_i y_i}{\sum M_i {y_i}^2}$$

$$\omega^2 = \frac{9.81\,[50(70.0) + 30(0.43)] \text{ x } 10^{-3}}{\left[50(70.0)^2 + 30(0.43)^2\right] \text{x } 10^{-6}}$$

$$\omega^2 = 15639.79$$

Therefore $\omega = 125.06 \text{ rad/sec}$ or 19.90 Hz

Example: By Dunkerley Method

The same problem in Figure 5 is considered. The natural frequencies of the 50 kg disk and the 30 kg disk are 126.27 rad/sec and 244.52 rad/sec, respectively. The natural frequency values are calculated using developed software.

Solution: We have the following principle for the Dunkerley Method,

$$\frac{1}{\omega^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2}$$

$$\frac{1}{\omega^2} = \frac{1}{126.27^2} + \frac{1}{244.52^2}$$

$$\omega^2 = 12587.44 \ \ \text{rad}^2/\text{sec}^2$$

Therefore $\omega = 112.19 \ \ \text{rad/sec}$ or 17.86 Hz

Example: By Transfer Matrix Method

The same problem in Figure 5 is considered for this method.

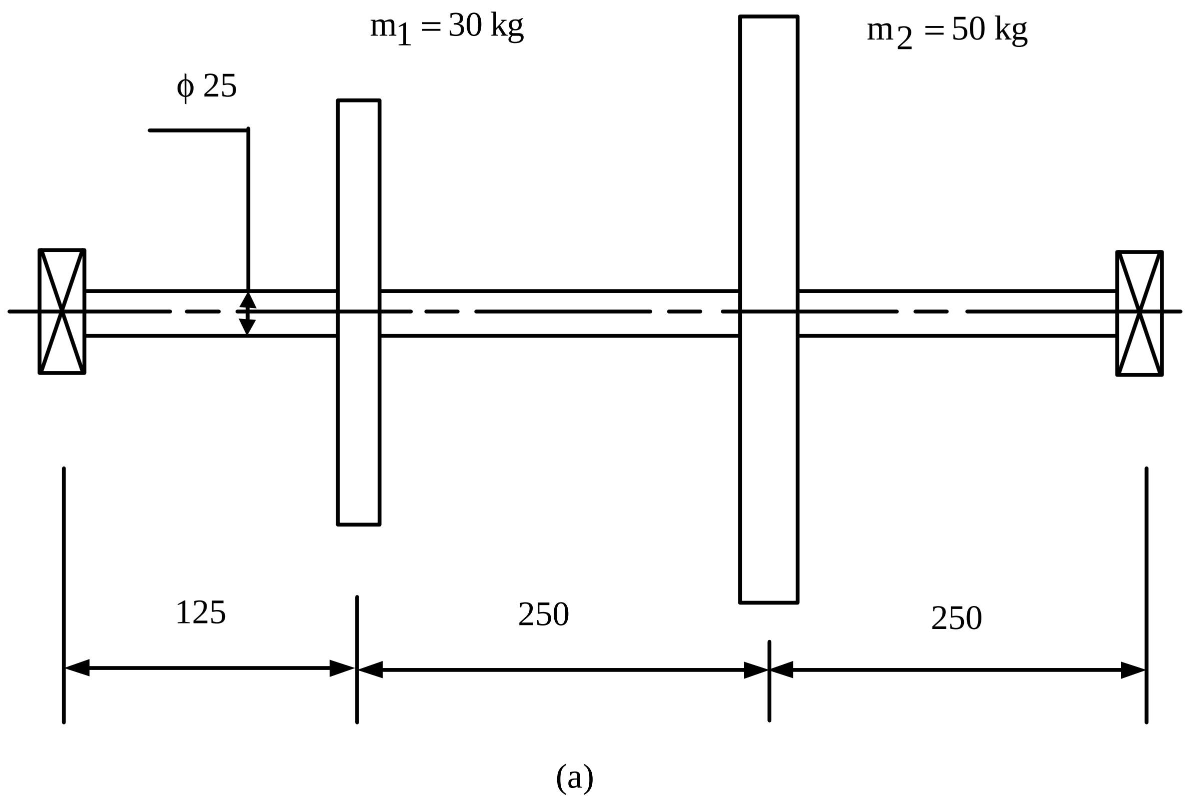
Solution: We have the following principle for the Transfer Matrix Method, using the following algorithm, we have developed a computer program using scientific language.

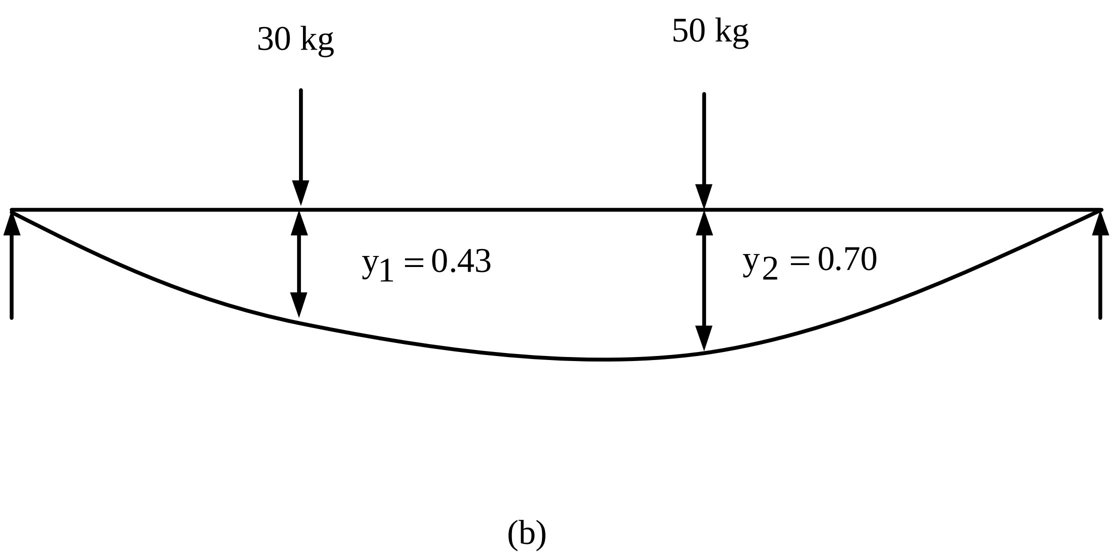
Transfer Matrix Algorithm

1. Take the trial value of the natural frequency ω.
2. Assign the element in the matrix with different variable.
3. Form the mass and field matrix.
4. Arrange the matrix in the order of multiplication and multiply it.
5. Apply boundary conditions to the overall transfer matrix.
6. Find the determinant value. If it is zero or nearly zero then its ω will be the natural frequency. Otherwise repeat the same procedure for a different value of ω.

The following natural frequency was calculated via a computer program:

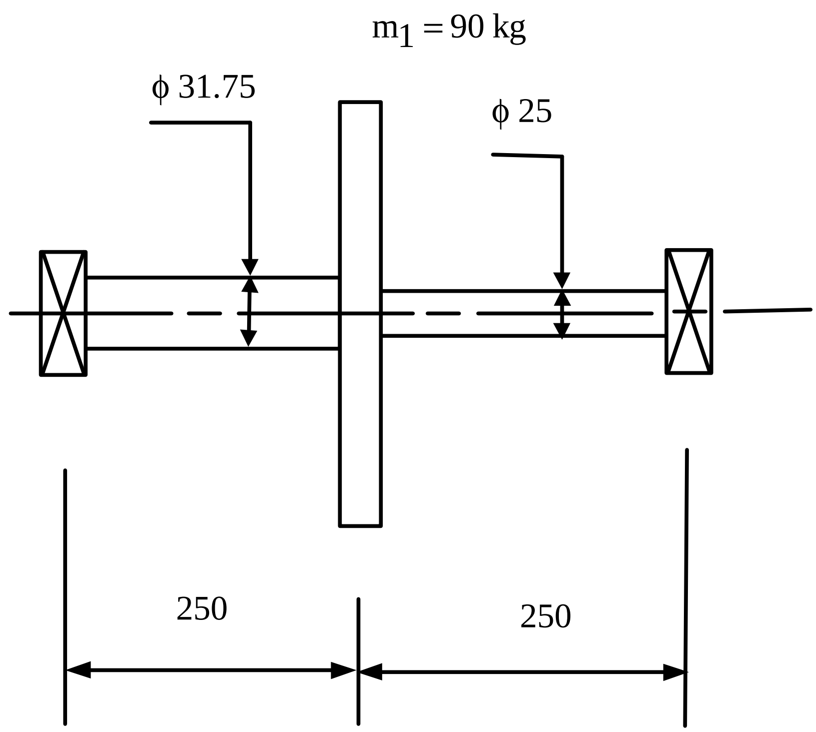
$$\omega = 115.25 \ \ \text{rad/sec} \qquad \text{or} \quad 18.34 \ \ \text{Hz}$$





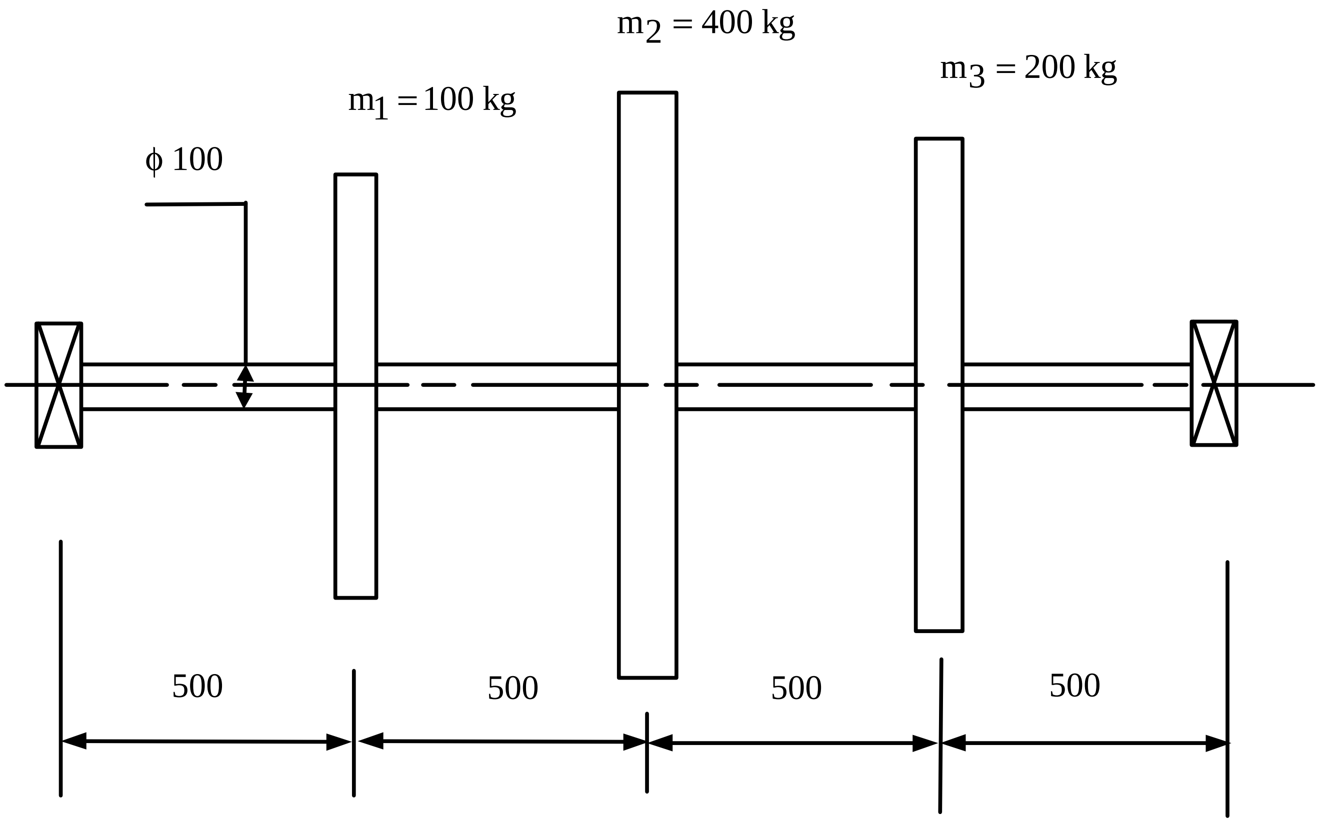
Note: All dimensions are in mm unless otherwise specified.

Figure 5. Example 1. Two disk and shaft system



Note: All dimensions are in mm unless otherwise specified.

Figure 6. Example 2. Step shaft and disk system



Note: All dimensions are in mm unless otherwise specified.

Figure 7. Example 3. Three disk and shaft system

Summary

A comparison of the results is given in Table 1.

Table 1.
Comparison of Natural Frequencies using Transfer Matrix Method with other Methods

Approach	Transfer Matrix Method f_n (Hz)	Dunkerley Method f_n (Hz)	Rayleigh Method f_n (Hz)
Example 1	18.34	17.86	19.90
Example 2	24.28	-	24.66
Example 3	16.52	16.17	-

Conclusion

There are a number of methods to find the lateral natural frequencies of undamped systems. Simple methods suitable for hand calculations and the transfer matrix method were presented in this paper. Dunkerley's method and the Rayleigh method serve to illustrate the hand method for estimating the fundamental frequency.

Dunkerley's equation assumes that the fundamental frequency is much lower than the harmonics. By neglecting the harmonics, the estimated fundamental frequency is always lower than the actual value.

The Rayleigh method assumes a mode shape for the vibration at the natural frequency. The method is generally used to find the fundamental frequency because it is more difficult to estimate the mode shapes of the harmonics. If the assumed mode shape is not exact, it is equivalent to having additional constraints in the system. Hence the estimated frequency tends to be higher than the true value.

In the Holzer method, a trial frequency is assumed and the solution is achieved if the boundary conditions are satisfied. Transfer matrix techniques may be regarded as an extension of the Holzer method. A state vector is an array of number, each of which is the value of a variable at a given station in the system. Hence it describes the state of a system. A transfer matrix relates the state vector of one station to the next. Thus a recurrence formula is obtained. Hence more complex problems (like a Fiberizer in the sugar industry or a crusher in cement industry which have a number of disks in between the supports) can be handled by the transfer matrix method. A computer is required for this method, however. The program can be extended to the overhang of a shaft and the mode shape.

References

1. W.T. Thomson. *Mechanical Vibration*. George Allen and Unwin Ltd., 1950.
2. F.S. Tse, I.E. Morse, and R.T. Hinkle. *Mechanical Vibration Theory and Application*. Allyn and Bacon Inc., 1978.
3. J.S. Rao. *Rotor Dynamics*. Wiley Eastern Ltd., 1983.
4. P. Srinivasan. *Mechanica Vibration and Analysis*. Tata McGraw-Hill Co., 1982.