SEMI-DEFINITE SYSTEM EXAMPLES
Revision B

By Tom Irvine
Email: tomirvine@aol.com

October 24, 2011

A semi-definite system is a system that has a rigid-body mode as well as an elastic body mode. The rigid-body frequency is zero.

Consider two masses connected by a spring.

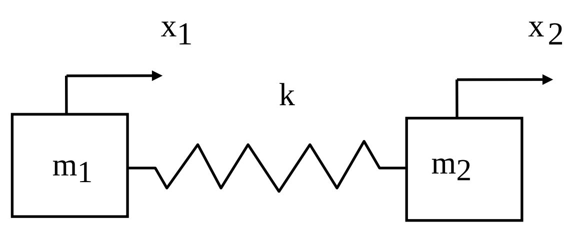


Figure 1.

The kinetic energy is

$$T = \frac{1}{2} m_1 \dot{x}_1{}^2 + \frac{1}{2} m_2 \dot{x}_2{}^2 \tag{1}$$

The potential energy is

$$U = \frac{1}{2} k (x_1 - x_2)^2 \tag{2}$$

$$\frac{d}{dt}\{T + U\} = 0 \tag{3}$$

$$\frac{d}{dt}\left\{\frac{1}{2} m_1 \dot{x}_1{}^2 + \frac{1}{2} m_2 \dot{x}_2{}^2 + \frac{1}{2} k (x_1 - x_2)^2\right\} = 0 \tag{4}$$

$$m_1\dot{x}_1\ddot{x}_1 + m_2\dot{x}_2\ddot{x}_2 + k(x_1 - x_2)\dot{x}_1 - k(x_1 - x_2)\dot{x}_2 = 0 \tag{5}$$

$$\{m_1\ddot{x}_1 + k(x_1 - x_2)\}\dot{x}_1 + \{m_2\ddot{x}_2 - k(x_1 - x_2)\}\dot{x}_2 = 0 \tag{6}$$

Equation (6) yields two equations.

$$\{m_1\ddot{x}_1 + k(x_1 - x_2)\}\dot{x}_1 = 0 \tag{7}$$

$$\{m_2\ddot{x}_2 - k(x_1 - x_2)\}\dot{x}_2 = 0 \tag{8}$$

Divide through by the respective velocity terms

$$m_1\ddot{x}_1 + k(x_1 - x_2) = 0 \tag{9}$$

$$m_2\ddot{x}_2 - k(x_1 - x_2) = 0 \tag{10}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11}$$

Seek a solution of the form

$$\bar{x} = \bar{q}\exp(j\omega t) \tag{12}$$

The q vector is the generalized coordinate vector.

Note that

$$\dot{\bar{x}} = j\omega\,\bar{q}\exp(j\omega t) \tag{13}$$

$$\ddot{\bar{x}} = -\omega^2\,\bar{q}\exp(j\omega t) \tag{14}$$

By substitution

$$-\omega^2 M\,\bar{q}\exp(j\omega t) + K\bar{q}\exp(j\omega t) = \bar{0} \tag{15}$$

$$\left\{-\omega^2 M \bar{q} + K \bar{q}\right\} \exp(j\omega t) = \bar{0} \tag{16}$$

$$-\omega_n{}^2 M \bar{q} + K \bar{q} = \bar{0} \tag{17}$$

$$\left\{-\omega^2 M + K\right\} \bar{q} = \bar{0} \tag{18}$$

$$\det \left\{ K - \omega^2 M \right\} = 0 \tag{19}$$

$$\det\left\{\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right\} = 0 \tag{20}$$

$$\left(k - \omega^2 m_1\right)\left(k - \omega^2 m_2\right) - k^2 = 0 \tag{21}$$

$$k^2 - k\omega^2 (m_1 + m_2) + \omega^4 m_1 m_2 - k^2 = 0 \tag{22}$$

$$-k\omega^2 (m_1 + m_2) + \omega^4 m_1 m_2 = 0 \tag{23}$$

$$\omega^2 \left[-k(m_1 + m_2) + \omega^2 m_1 m_2\right] = 0 \tag{24}$$

Thus the first root is

$$\omega_1 = 0 \tag{25}$$

$$f_1 = 0 \tag{26}$$

Find the second root

$$\left[-k(m_1 + m_2) + \omega^2 m_1 m_2\right] = 0 \tag{27}$$

$$\omega^2 = \frac{k(m_1 + m_2)}{m_1 m_2} \tag{28}$$

$$\omega_2 = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}} \tag{29}$$

$$f_2 = \frac{1}{2\pi} \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}} \tag{30}$$

Prescribed Motion

Calculate the response of the system for a prescribed acceleration $\ddot{x}_1$.

Let

$$z = x_2 - x_1 \tag{31}$$

Substitute equation (31) into (11).

$$\begin{bmatrix} m_1 & 0 \\ m_2 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{z} \end{bmatrix} + k \begin{bmatrix} -z \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{32}$$

The equation of motion is thus

$$m_2 \ddot{z} + kz = -m_2 \ddot{x}_1 \tag{33}$$

$$\ddot{z} + \left(\frac{k}{m_2} \right) z = -\ddot{x}_1 \tag{34}$$

Solve for z and $\ddot{z}$.

Then solve for $\ddot{x}_2$ using equation (31).

References

1. T. Irvine, The Energy Method, Rev C, Vibrationdata, 2002.
2. T. Irvine, Spring Surge Natural Frequencies, Vibrationdata, 2007.

APPENDIX A

Repeat the example in the main text but also consider the mass of the spring.

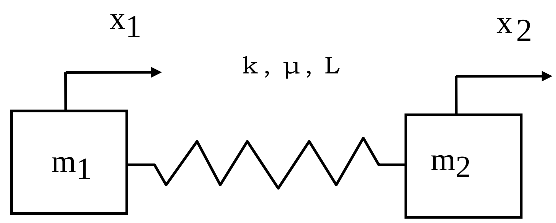


Figure A-1.

The kinetic energy of the spring is found in the following steps. Define a local variable ξ which is a measure of the distance along the spring.

$$0 \leq \xi \leq L \tag{A-1}$$

The velocity at any point on the spring is thus

$$\dot{x}_1\left(\frac{L-\xi}{L}\right) + \dot{x}_2 \frac{\xi}{L} \tag{A-2}$$

Now divide the spring into n segments. The kinetic energy of the spring is thus

$$KE_{spring} = \frac{1}{2}\sum_{i=1}^{n}\left\{\left[\dot{x}_1\left(\frac{L-\xi}{L}\right) + \dot{x}_2 \frac{\xi}{L}\right]^2 \mu\,\Delta\xi\right\} \tag{A-3}$$

Take the limit as n approaches infinity.

$$KE_{spring} = \frac{1}{2}\int_0^L \left[\dot{x}_1 \left(\frac{L-\xi}{L} \right) + \dot{x}_2 \frac{\xi}{L} \right]^2 \mu \, d\xi \qquad \text{(A-4)}$$

$$KE_{spring} = \frac{1}{2}\int_0^L \left[\dot{x}_1{}^2 \left(\frac{L-\xi}{L} \right)^2 + 2\dot{x}_1 \dot{x}_2 \frac{\xi}{L}\left(\frac{L-\xi}{L} \right) + \dot{x}_2{}^2 \left(\frac{\xi}{L} \right)^2 \right] \mu \, d\xi \qquad \text{(A-5)}$$

$$KE_{spring} =$$

$$\frac{1}{2}\int_0^L \left[\dot{x}_1{}^2 \left(\frac{L-\xi}{L} \right)^2 \right] \mu \, d\xi$$

$$+ \frac{1}{2}\int_0^L \left[2\dot{x}_1 \dot{x}_2 \frac{\xi}{L}\left(\frac{L-\xi}{L} \right) \right] \mu \, d\xi$$

$$+ \frac{1}{2}\int_0^L \left[\dot{x}_2{}^2 \left(\frac{\xi}{L} \right)^2 \right] \mu \, d\xi$$

(A-6)

$$KE_{spring} =$$

$$\frac{1}{2}\dot{x}_1{}^2\mu\int_0^L\left(1-\frac{2\xi}{L}+\frac{\xi^2}{L^2}\right)d\xi$$

$$+\dot{x}_1\,\dot{x}_2\,\mu\int_0^L\left(\frac{\xi}{L}-\frac{\xi^2}{L^2}\right)d\xi$$

$$+\frac{1}{2}\dot{x}_2{}^2\mu\int_0^L\left(\frac{\xi}{L}\right)^2 d\xi$$

(A-7)

$$KE_{spring} =$$

$$\frac{1}{2}\dot{x}_1{}^2\mu\left(\xi-\frac{\xi^2}{L}+\frac{\xi^3}{3L^2}\right)\Bigg|_0^L$$

$$+\dot{x}_1\,\dot{x}_2\,\mu\left(\frac{\xi^2}{2L}-\frac{\xi^3}{3L^2}\right)\Bigg|_0^L$$

$$+\frac{1}{2}\dot{x}_2{}^2\mu\left(\frac{\xi^3}{3L^2}\right)\Bigg|_0^L$$

(A-8)

$$KE_{spring} = \frac{1}{6}\mu L\left(\dot{x}_1{}^2+\dot{x}_1\,\dot{x}_2\;+\dot{x}_2{}^2\right) \qquad \text{(A-9)}$$

$$\frac{d}{dt}\left\{\frac{1}{2}m_1\dot{x}_1{}^2+\frac{1}{2}m_2\dot{x}_2{}^2+\frac{1}{6}\mu L\left(\dot{x}_1{}^2+\dot{x}_1\dot{x}_2+\dot{x}_2{}^2\right)+\frac{1}{2}k(x_1-x_2)^2\right\}=0 \qquad \text{(A-10)}$$

$$m_1\dot{x}_1\ddot{x}_1+m_2\dot{x}_2\ddot{x}_2+\mu L\left(\frac{1}{3}\dot{x}_1\ddot{x}_1+\frac{1}{6}\dot{x}_2\ddot{x}_1+\frac{1}{6}\dot{x}_1\ddot{x}_2+\frac{1}{3}\dot{x}_2\ddot{x}_2\right)$$
$$+k(x_1-x_2)\dot{x}_1-k(x_1-x_2)\dot{x}_2=0$$
(A-11)

$$\left\{m_1\ddot{x}_1+\mu L\left(\frac{1}{3}\ddot{x}_1+\frac{1}{6}\ddot{x}_2\right)+k(x_1-x_2)\right\}\dot{x}_1$$
$$+\left\{m_2\ddot{x}_2+\mu L\left(\frac{1}{6}\ddot{x}_1+\frac{1}{3}\ddot{x}_2\right)-k(x_1-x_2)\right\}\dot{x}_2=0$$
(A-12)

Equation (A-12) yields two equations.

$$\left\{m_1\ddot{x}_1+\mu L\left(\frac{1}{3}\ddot{x}_1+\frac{1}{6}\ddot{x}_2\right)+k(x_1-x_2)\right\}\dot{x}_1=0 \qquad \text{(A-13)}$$

$$\left\{m_2\ddot{x}_2+\mu L\left(\frac{1}{6}\ddot{x}_1+\frac{1}{3}\ddot{x}_2\right)-k(x_1-x_2)\right\}\dot{x}_2=0 \qquad \text{(A-14)}$$

$$m_1\ddot{x}_1+\mu L\left(\frac{1}{3}\ddot{x}_1+\frac{1}{6}\ddot{x}_2\right)+k(x_1-x_2)=0 \qquad \text{(A-15)}$$

$$m_2\ddot{x}_2+\mu L\left(\frac{1}{6}\ddot{x}_1+\frac{1}{3}\ddot{x}_2\right)-k(x_1-x_2)=0 \qquad \text{(A-16)}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 + \frac{1}{3}\mu L & \frac{1}{6}\mu L \\ \frac{1}{6}\mu L & m_2 + \frac{1}{3}\mu L \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{(A-17)}$$

$$\det \left\{ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 + \frac{1}{3}\mu L & \frac{1}{6}\mu L \\ \frac{1}{6}\mu L & m_2 + \frac{1}{3}\mu L \end{bmatrix} \right\} = 0 \quad \text{(A-18)}$$

$$\left[m_{11} m_{22} - m_{12} m_{21} \right] \omega^4 + \left[-m_{11} k_{22} - m_{22} k_{11} + m_{12} k_{21} + m_{21} k_{12} \right] \omega^2$$
$$+ \left[k_{11} k_{22} - k_{12} k_{21} \right] = 0$$

(A-19)

$$\left[\left(m_1 + \frac{1}{3}\mu L \right)\left(m_2 + \frac{1}{3}\mu L \right) - \frac{1}{36}\mu^2 L^2 \right] \omega^4$$
$$+ k\left[-m_1 - \frac{1}{3}\mu L - m_2 - \frac{1}{3}\mu L - \frac{1}{6}\mu L - \frac{1}{6}\mu L \right] \omega^2 = 0$$

(A-20)

Thus the first root is

$$\omega_1 = 0 \quad \text{(A-21)}$$

$$f_1 = 0 \quad \text{(A-22)}$$

$$\left[\left(m_1+\frac{1}{3}\mu L\right)\left(m_2+\frac{1}{3}\mu L\right)-\frac{1}{36}\mu^2 L^2\right]\omega^2+k\left[-m_1-m_2-\mu L\right] = 0 \qquad \text{(A-23)}$$

$$\omega_2=\sqrt{\frac{k(m_1+m_2+\mu L)}{\left(m_1+\frac{1}{3}\mu L\right)\left(m_2+\frac{1}{3}\mu L\right)-\frac{1}{36}\mu^2 L^2}} \qquad \text{(A-24)}$$

$$f_2=\frac{1}{2\pi}\sqrt{\frac{k(m_1+m_2+\mu L)}{m_1 m_2+\frac{1}{3}\mu L(m_1+m_2)+\frac{1}{12}\mu^2 L^2}} \qquad \text{(A-25)}$$

Special Case

Consider the case where both masses equal zero.

$$f_2=\frac{1}{2\pi}\sqrt{\frac{12k}{\mu L}} \approx \quad 0.551\sqrt{\frac{k}{\mu L}} \qquad \text{(A-26)}$$

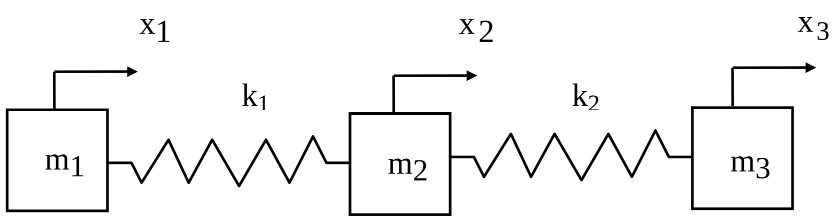
The “exact” frequency per Reference 2 is

$$f_2 = \quad 0.500\sqrt{\frac{k}{\mu L}} \qquad \text{(A-27)}$$

The energy method thus-over predicts the frequency by 10%. The energy method could be improved by modeling the spring with a higher-order interpolation or with additional nodes.

APPENDIX B

Consider a three-degree-of-freedom system with mass-less springs.



The kinetic energy is

$$T = \frac{1}{2} m_1 \dot{x}_1{}^2 + \frac{1}{2} m_2 \dot{x}_2{}^2 + \frac{1}{2} m_3 \dot{x}_3{}^2 \tag{B-1}$$

The potential energy is

$$U = \frac{1}{2} k_1 (x_1 - x_2)^2 + \frac{1}{2} k_2 (x_2 - x_3)^2 \tag{B-2}$$

$$\frac{d}{dt}\{T + U\} = 0 \tag{B-3}$$

$$\frac{d}{dt}\left\{\frac{1}{2} m_1 \dot{x}_1{}^2 + \frac{1}{2} m_2 \dot{x}_2{}^2 + \frac{1}{2} m_3 \dot{x}_3{}^2 + \frac{1}{2} k_1 (x_1 - x_2)^2 + \frac{1}{2} k_2 (x_2 - x_3)^2\right\} = 0 \tag{B-4}$$

$$\begin{aligned} & m_1 \dot{x}_1 \ddot{x}_1 + m_2 \dot{x}_2 \ddot{x}_2 + m_3 \dot{x}_3 \ddot{x}_3 \\ & + k_1 (x_1 - x_2) \dot{x}_1 - k_1 (x_1 - x_2) \dot{x}_2 + k_2 (x_2 - x_3) \dot{x}_2 - k_2 (x_2 - x_3) \dot{x}_3 = 0 \end{aligned} \tag{B-5}$$

$$\{m_1\ddot{x}_1 + k_1(x_1 - x_2)\}\dot{x}_1 + \{m_2\ddot{x}_2 - k_1(x_1 - x_2) + k_2(x_2 - x_3)\}\dot{x}_2 + \{m_3\ddot{x}_3 - k_2(x_2 - x_3)\}\dot{x}_3 = 0 \quad \text{(B-6)}$$

Equation (B-6) yields three equations.

$$\{m_1\ddot{x}_1 + k_1(x_1 - x_2)\}\dot{x}_1 = 0 \quad \text{(B-7)}$$

$$\{m_2\ddot{x}_2 - k_1(x_1 - x_2) + k_2(x_2 - x_3)\}\dot{x}_2 = 0 \quad \text{(B-8)}$$

$$\{m_3\ddot{x}_3 - k_2(x_2 - x_3)\}\dot{x}_3 = 0 \quad \text{(B-9)}$$

Divide through by the respective velocity terms

$$m_1\ddot{x}_1 + k_1(x_1 - x_2) = 0 \quad \text{(B-10)}$$

$$m_2\ddot{x}_2 - k_1(x_1 - x_2) + k_2(x_2 - x_3) = 0 \quad \text{(B-11)}$$

$$m_3\ddot{x}_3 - k_3(x_2 - x_3) = 0 \quad \text{(B-12)}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(B-13)}$$