Consider a beam which undergoes shear displacement only.

\[ u(x, t) = \text{Transverse displacement} \]
\[ G = \text{Shear modulus} \]
\[ A = \text{Cross-section area} \]
\[ k = \text{Shear factor} \]
\[ \rho = \text{Mass/volume} \]

Assume a uniform cross-section and mass density.

The transverse shear displacement \( u(x, t) \) is governed by the equation

\[ \rho A \frac{\partial^2 u}{\partial t^2} = kGA \frac{\partial^2 u}{\partial x^2} \] (1)

Equation (1) is taken from Reference 1.

\[ \rho \frac{\partial^2 u}{\partial t^2} = kG \frac{\partial^2 u}{\partial x^2} \] (2)
\[ \frac{\partial^2 u}{\partial t^2} = \left[ \frac{kG}{\rho} \right] \frac{\partial^2 u}{\partial x^2} \]  

(3)

Separate the variables. Let

\[ u(x,t) = U(x)T(t) \]  

(4)

By substitution,

\[ \left[ \frac{kG}{\rho} \right] \frac{\partial^2}{\partial x^2} [U(x)T(t)] = \frac{\partial^2}{\partial t^2} [U(x)T(t)] \]  

(5)

Perform the partial differentiation.

\[ \left[ \frac{kG}{\rho} \right] U''(x)T(t) = U(x)T''(t) \]  

(6)

Divide through by \( U(x)T(t) \).

\[ \left[ \frac{kG}{\rho} \right] \frac{U''(x)}{U(x)} = \frac{T''(t)}{T(t)} \]  

(7)

\[ \left[ \frac{kG}{\rho} \right] \frac{U''(x)}{U(x)} = \frac{T''(t)}{T(t)} \]  

(8)

Each side of equation (8) must equal a constant. Let \( \omega \) be a constant.

\[ \left[ \frac{kG}{\rho} \right] \frac{U''(x)}{U(x)} = \frac{T''(t)}{T(t)} = -\omega^2 \]  

(9)

The time equation is

\[ \frac{T''(t)}{T(t)} = -\omega^2 \]  

(10)
\[ T''(t) = -\omega^2 T(t) \]  
(11) \[ T''(t) + \omega^2 T(t) = 0 \]  
(12)

Propose a solution

\[ T(t) = a \sin(\omega t) + b \cos(\omega t) \]  
(13)

\[ T'(t) = a \omega \cos(\omega t) - b \omega \sin(\omega t) \]  
(14)

\[ T''(t) = -a \omega^2 \sin(\omega t) - b \omega^2 \cos(\omega t) \]  
(15)

Verify the proposed solution. Substitute into equation (12).

\[
-a \omega^2 \sin(\omega t) - b \omega^2 \cos(\omega t) + \omega^2 \left[ \sin(\omega t) + \omega^2 \cos(\omega t) \right] = 0
\]  
(16)

\[ 0 = 0 \]  
(17)

Equation (17) is thus verified as a solution.

There is not a unique \( \omega \), however, in equation (16). This is demonstrated later in the derivation. Thus a subscript \( n \) must be added as follows.

\[ T_n(t) = a_n \sin(\omega_n t) + b_n \cos(\omega_n t) \]  
(18)

The spatial equation is

\[
\left[ \frac{kG}{\rho} \right] U''(x) - \omega^2 = 0
\]  
(19)

Let

\[ c = \sqrt{\frac{kG}{\rho}} \]  
(20)

\[ U''(x) = -\frac{\omega^2}{c^2} U(x) \]  
(21a)
\[ U''(x) + \frac{\omega^2}{c^2} U(x) = 0 \quad (21b) \]

The displacement solution is

\[ U(x) = d \sin \left( \frac{\omega x}{c} \right) + e \cos \left( \frac{\omega x}{c} \right) \quad (22) \]

The slope equation is

\[ U'(x) = \left[ \frac{\omega}{c} \right] \left[ dc \cos \left( \frac{\omega x}{c} \right) - es \sin \left( \frac{\omega x}{c} \right) \right] \quad (23) \]

The solutions for various boundary condition cases are given in the appendices.

Reference

APPENDIX A

Case I. Fixed-Free
The left boundary conditions is

\[ u(0, t) = 0 \quad \text{(zero displacement)} \quad \text{(A-1)} \]
\[ U(0)T(t) = 0 \quad \text{(A-2)} \]
\[ U(0) = 0 \quad \text{(A-3)} \]

The right boundary condition is

\[ \frac{\partial}{\partial x} u(x, t) \bigg|_{x=L} = 0 \quad \text{(zero stress)} \quad \text{(A-4)} \]
\[ U'(L)T(t) = 0 \quad \text{(A-5)} \]
\[ U''(L) = 0 \quad \text{(A-6)} \]

Substitute equation (A-3) into (22).

\[ e = 0 \quad \text{(A-7)} \]

Thus, the displacement equation becomes

\[ U(x) = d \sin \left( \frac{\omega x}{c} \right) \quad \text{(A-8)} \]
\[ U'(x) = \left[ \frac{\omega}{c} \right] \left[ d \cos \left( \frac{\omega x}{c} \right) \right] \quad \text{(A-9)} \]

Substitute equation (A-6) into equation (A-9).

\[ d \cos \left( \frac{\omega L}{c} \right) = 0 \quad \text{(A-10)} \]
The constant \( d \) must be non-zero for a non-trivial solution. Thus,

\[
\frac{\omega_n L}{c} = \left(\frac{2n-1}{2}\right)\pi, \quad n = 1,2,3,...
\] (A-11)

\[
\omega_n = \left(\frac{2n-1}{2}\right)\pi \frac{c}{L}, \quad n = 1,2,3,...
\] (A-12)

The \( \omega \) term is given a subscript \( n \) because there are multiple roots.

\[
\omega_n = \left(\frac{2n-1}{2}\right)\pi \sqrt{\frac{kG}{\rho L}}, \quad n = 1,2,3,...
\] (A-13)

The displacement function for the fixed-free beam is

\[
U_n(x) = d_n \sin\left(\frac{\omega_n x}{c}\right)
\] (A-14)

\[
U_n(x) = d_n \sin\left(\frac{(2n-1)\pi x}{2L}\right)
\] (A-15)
Case II. Fixed-Fixed

The left boundary condition is

\[ u(0, t) = 0 \quad \text{(zero displacement)} \]  \hspace{1cm} (B-1)
\[ U(0)T(t) = 0 \]  \hspace{1cm} (B-2)
\[ U(0) = 0 \]  \hspace{1cm} (B-3)

The right boundary condition is

\[ u(L, t) = 0 \quad \text{(zero displacement)} \]  \hspace{1cm} (B-4)
\[ U(L)T(t) = 0 \]  \hspace{1cm} (B-5)
\[ U(L) = 0 \]  \hspace{1cm} (B-6)

Substitute equation (B-3) into (22).

\[ e = 0 \]  \hspace{1cm} (B-7)

Thus, the displacement equation becomes

\[ U(x) = d \sin \left( \frac{\omega x}{c} \right) \]  \hspace{1cm} (B-8)

Substitute equation (B-6) into (B-8).

\[ d \sin \left( \frac{\omega L}{c} \right) = 0 \]  \hspace{1cm} (B-9)

The constant \( d \) must be non-zero for a non-trivial solution. Thus,

\[ \frac{\omega L}{c} = n\pi, \quad n = 1, 2, 3, \ldots \]  \hspace{1cm} (B-10)

The \( \omega \) term is given a subscript \( n \) because there are multiple roots.
\[ \omega_n = n \pi \frac{c}{L}, \quad n = 1, 2, 3, \ldots \quad \text{(B-11)} \]

The displacement function for the fixed-fixed beam is

\[ U_n(x) = d_n \sin \left( \frac{\omega_n x}{c} \right) \quad \text{(B-12)} \]

\[ U_n(x) = d_n \sin \left( \frac{n\pi x}{L} \right) \quad \text{(B-13)} \]