NATURAL FREQUENCIES OF A SHEAR BEAM

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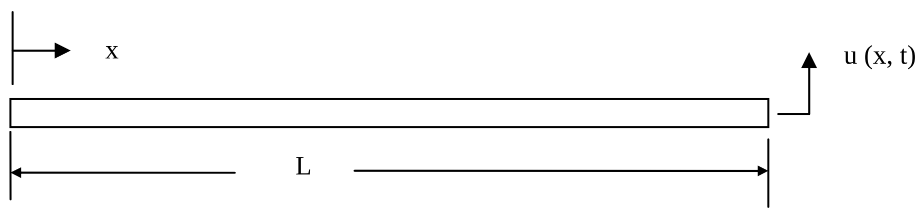
Consider a beam which undergoes shear displacement only.



u(x, t)	=	Transverse displacement
G	=	Shear modulus
A	=	Cross-section area
k	=	Shear factor
ρ	=	Mass/volume

Assume a uniform cross-section and mass density.

The transverse shear displacement u(x, t) is governed by the equation

$$\rho A \frac{\partial^2 u}{\partial t^2} = kGA \frac{\partial^2 u}{\partial x^2} \tag{1}$$

Equation (1) is taken from Reference 1.

$$\rho \frac{\partial^2 u}{\partial t^2} = kG \frac{\partial^2 u}{\partial x^2} \tag{2}$$

$$\frac{\partial^2 u}{\partial t^2} = \left[\frac{kG}{\rho}\right]\frac{\partial^2 u}{\partial x^2} \tag{3}$$

Separate the variables. Let

$$u(x,t) = U(x)T(t) \tag{4}$$

By substitution,

$$\left[\frac{kG}{\rho}\right]\frac{\partial^2}{\partial x^2}\left[U(x)T(t)\right] = \frac{\partial^2}{\partial t^2}\left[U(x)T(t)\right] \tag{5}$$

Perform the partial differentiation.

$$\left[\frac{kG}{\rho}\right]U''(x)T(t) = U(x)T''(t) \tag{6}$$

Divide through by U(x)T(t).

$$\left[\frac{kG}{\rho}\right]\frac{U''(x)}{U(x)} = \frac{T''(t)}{T(t)} \tag{7}$$

$$\left[\frac{kG}{\rho}\right]\frac{U''(x)}{U(x)} = \frac{T''(t)}{T(t)} \tag{8}$$

Each side of equation (8) must equal a constant. Let ω be a constant.

$$\left[\frac{kG}{\rho}\right]\frac{U''(x)}{U(x)} = \frac{T''(t)}{T(t)} = -\omega^2 \tag{9}$$

The time equation is

$$\frac{T''(t)}{T(t)} = -\omega^2 \tag{10}$$

$$T''(t) = -\omega^2 T(t) \tag{11}$$

$$T''(t) + \omega^2 T(t) = 0 \tag{12}$$

Propose a solution

$$T(t) = a \sin(\omega t) + b \cos(\omega t) \tag{13}$$

$$T'(t) = a\,\omega \cos(\omega t) - b\,\omega \sin(\omega t) \tag{14}$$

$$T''(t) = -a\,\omega^2 \sin(\omega t) - b\,\omega^2 \cos(\omega t) \tag{15}$$

Verify the proposed solution. Substitute into equation (12).

$$-a\,\omega^2 \sin(\omega t) - b\,\omega^2 \cos(\omega t) + \omega^2\left[\sin(\omega t) + \omega^2 \cos(\omega t)\right] = 0 \tag{16}$$

$$0 = 0 \tag{17}$$

Equation (17) is thus verified as a solution.

There is not a unique ω, however, in equation (16). This is demonstrated later in the derivation. Thus a subscript n must be added as follows.

$$T_n(t) = a_n \sin(\omega_n t) + b_n \cos(\omega_n t) \tag{18}$$

The spatial equation is

$$\left[\frac{kG}{\rho}\right]\frac{U''(x)}{U(x)} = -\omega^2 \tag{19}$$

Let

$$c = \sqrt{\frac{kG}{\rho}} \tag{20}$$

$$U''(x) = -\frac{\omega^2}{c^2}\,U(x) \tag{21a}$$

$$U''(x) + \frac{\omega^2}{c^2} U(x) = 0 \tag{21b}$$

The displacement solution is

$$U(x) = d \sin\left(\frac{\omega x}{c}\right) + e \cos\left(\frac{\omega x}{c}\right) \tag{22}$$

The slope equation is

$$U'(x) = \left[\frac{\omega}{c}\right]\left[d\cos\left(\frac{\omega x}{c}\right) - e\sin\left(\frac{\omega x}{c}\right)\right] \tag{23}$$

The solutions for various boundary condition cases are given in the appendices.

Reference

1. J. Betbeder-Matibet, Closed-form Solutions for SRSS Response of Shear Beams, Proceedings of the Tenth World Conference on Earthquake Engineering, 1992.

APPENDIX A

Case I. Fixed-Free

The left boundary conditions is

$$u(0,t) = 0 \qquad \text{(zero displacement)} \tag{A-1}$$

$$U(0)T(t) = 0 \tag{A-2}$$

$$U(0) = 0 \tag{A-3}$$

The right boundary condition is

$$\left.\frac{\partial}{\partial x} u(x,t)\right|_{x=L} = 0 \qquad \text{(zero stress)} \tag{A-4}$$

$$U'(L)T(t) = 0 \tag{A-5}$$

$$U'(L) = 0 \tag{A-6}$$

Substitute equation (A-3) into (22).

$$e = 0 \tag{A-7}$$

Thus, the displacement equation becomes

$$U(x) = d\sin\left(\frac{\omega x}{c}\right) \tag{A-8}$$

$$U'(x) = \left[\frac{\omega}{c}\right]\left[d\cos\left(\frac{\omega x}{c}\right)\right] \tag{A-9}$$

Substitute equation (A-6) into equation (A-9).

$$d\cos\left(\frac{\omega L}{c}\right) = 0 \tag{A-10}$$

The constant d must be non-zero for a non-trivial solution. Thus,

$$\frac{\omega_n L}{c} = \left(\frac{2n-1}{2}\right)\pi, \quad n = 1,2,3,\ldots \tag{A-11}$$

$$\omega_n = \left(\frac{2n-1}{2}\right)\pi \frac{c}{L}, \quad n = 1,2,3,\ldots \tag{A-12}$$

The ω term is given a subscript n because there are multiple roots.

$$\omega_n = \left(\frac{2n-1}{2}\right)\pi \sqrt{\frac{kG}{\rho L}}, \quad n = 1,2,3,\ldots \tag{A-13}$$

The displacement function for the fixed-free beam is

$$U_n(x) = d_n \sin\left(\frac{\omega_n x}{c}\right) \tag{A-14}$$

$$U_n(x) = d_n \sin\left(\frac{(2n-1)\pi\, x}{2L}\right) \tag{A-15}$$

APPENDIX B

Case II. Fixed-Fixed

The left boundary condition is

$$u(0,t) = 0 \qquad \text{(zero displacement)} \tag{B-1}$$

$$U(0)T(t) = 0 \tag{B-2}$$

$$U(0) = 0 \tag{B-3}$$

The right boundary condition is

$$u(L,t) = 0 \qquad \text{(zero displacement)} \tag{B-4}$$

$$U(L)T(t) = 0 \tag{B-5}$$

$$U(L) = 0 \tag{B-6}$$

Substitute equation (B-3) into (22).

$$e = 0 \tag{B-7}$$

Thus, the displacement equation becomes

$$U(x) = d \sin\left(\frac{\omega x}{c}\right) \tag{B-8}$$

Substitute equation (B-6) into (B-8).

$$d \sin\left(\frac{\omega L}{c}\right) = 0 \tag{B-9}$$

The constant d must be non-zero for a non-trivial solution. Thus,

$$\frac{\omega_n L}{c} = n\pi, \quad n = 1, 2, 3, \ldots \tag{B-10}$$

The ω term is given a subscript n because there are multiple roots.

$$\omega_n = n\pi \frac{c}{L}, \quad n = 1, 2, 3, \ldots \tag{B-11}$$

The displacement function the fixed-fixed beam is

$$U_n(x) = d_n \sin\left(\frac{\omega_n x}{c}\right) \tag{B-12}$$

$$U_n(x) = d_n \sin\left(\frac{n\pi x}{L}\right) \tag{B-13}$$