VIBRATION OF A SHEAR FRAME BUILDING

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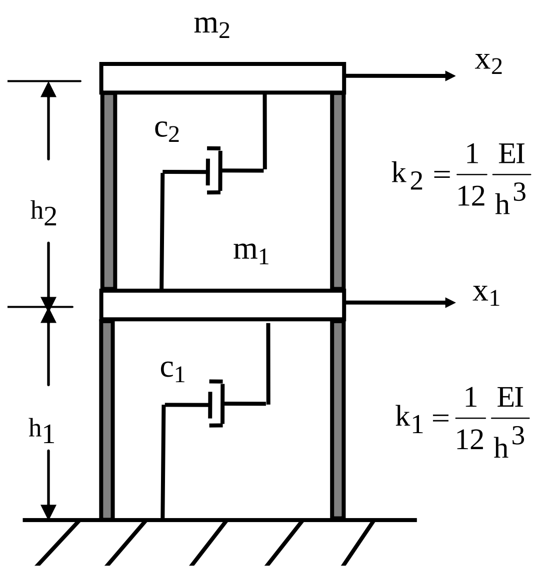

Figure 1.

E	Elastic modulus
I	Area moment of inertia
h	Story height
m	Mass
c	Damping coefficient
k	Stiffness

Consider the two-story shear frame system, as shown in Figure 1.

The slabs are rigid masses. The stiffness is taken from the column flexure.

The system can be modeled by the spring-mass system in Figure 2.

Free-body diagrams are shown in Figure 3.

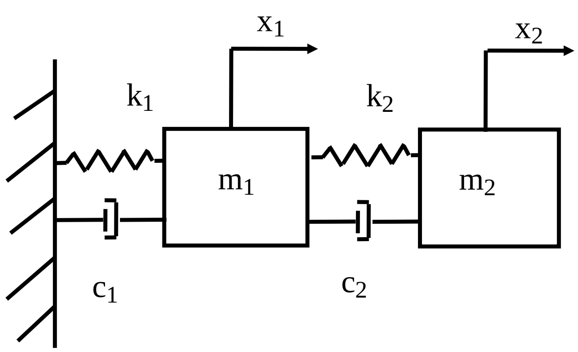

Figure 2.

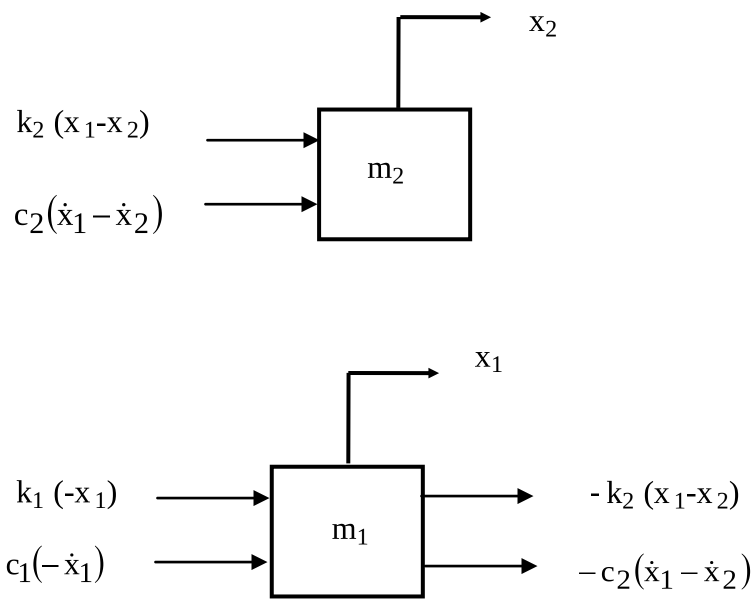

Figure 3.

Determine the equation of motion for mass 2.

$$\sum F = m_2 \ddot{x}_2 \tag{1}$$

$$m_2 \ddot{x}_2 = c_2(\dot{x}_1 - \dot{x}_2) + k_2(x_1 - x_2) \tag{2}$$

$$m_2 \ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = 0 \tag{3}$$

Determine the equation of motion for mass 1.

$$\sum F = m_1 \ddot{x}_1 \tag{4}$$

$$m_1 \ddot{x}_1 = -c_2(\dot{x}_1 - \dot{x}_2) + c_1(-\dot{x}_1) - k_2(x_1 - x_2) + k_1(-x_1) \tag{5}$$

$$m_1 \ddot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + c_1\dot{x}_1 + k_2(x_1 - x_2) + k_1 x_1 = 0 \tag{6}$$

$$m_1 \ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2 x_2 = 0 \tag{7}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8}$$

Represent as

$$M\ddot{\bar{x}} + C\dot{\bar{x}} + K\bar{x} = F \tag{9}$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{10}$$

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \tag{11}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \quad (12)$$

$$F = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (13)$$

Consider the undamped, homogeneous form of equation (9).

$$M \, \bar{\ddot{x}} + K \, \bar{x} = \bar{0} \quad (14)$$

Seek a solution of the form

$$\bar{x} = \bar{q} \exp(j\omega t) \quad (15)$$

The q vector is the generalized coordinate vector.

Note that

$$\bar{\dot{x}} = j\omega \, \bar{q} \exp(j\omega t) \quad (16)$$

$$\bar{\ddot{x}} = -\omega^2 \, \bar{q} \exp(j\omega t) \quad (17)$$

Substitute these equations into equation (14).

$$-\omega^2 M \, \bar{q} \exp(j\omega t) + K \, \bar{q} \exp(j\omega t) = \bar{0} \quad (18)$$

$$\left\{ -\omega^2 M + K \right\} \bar{q} \exp(j\omega t) = \bar{0} \quad (19)$$

$$\left\{ -\omega^2 M + K \right\} \bar{q} = \bar{0} \quad (20)$$

$$\left\{ K - \omega^2 M \right\} \bar{q} = \bar{0} \quad (21)$$

Equation (21) is an example of a generalized eigenvalue problem. The eigenvalues can be found by setting the determinant equal to zero.

$$\det\left\{ K - \omega^2 M \right\} = 0 \tag{22}$$

$$\det\left\{ \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right\} = 0 \tag{23}$$

$$\det\left\{ \begin{matrix} (k_1 + k_2) - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{matrix} \right\} = 0 \tag{24}$$

$$\left[(k_1 + k_2) - \omega^2 m_1\right]\left[k_2 - \omega^2 m_2\right] - k_2^{\,2} = 0 \tag{25}$$

$$-\omega^4 m_1 m_2 + \omega^2\left[-m_2(k_1 + k_2) - m_1 k_2\right] - k_2^{\,2} + k_2(k_1 + k_2) = 0 \tag{26}$$

$$-\omega^4 m_1 m_2 + \omega^2\left[-m_2(k_1 + k_2) - m_1 k_2\right] - k_2^{\,2} + k_1 k_2 + k_2^{\,2} = 0 \tag{27}$$

$$-\omega^4 m_1 m_2 + \omega^2\left[-m_2(k_1 + k_2) - m_1 k_2\right] + k_1 k_2 = 0 \tag{28}$$

The eigenvalues are the roots of the polynomial.

$$\omega_1^{\,2} = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{29}$$

$$\omega_2^{\,2} = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{30}$$

where

$$a = m_1 m_2 \tag{31}$$

$$b = \left[-m_2 (k_1 + k_2) - m_1 k_2 \right] \tag{32}$$

$$c = k_1 k_2 \tag{33}$$

The eigenvectors are found via the following equations.

$$\left\{ K - \omega_1^{\,2}\, M \right\} \bar{q}_1 = \bar{0} \tag{34}$$

$$\left\{ K - \omega_2^{\,2}\, M \right\} \bar{q}_2 = \bar{0} \tag{35}$$

where

$$\bar{q}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{36}$$

$$\bar{q}_2 = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{37}$$

An eigenvector matrix Q can be formed. The eigenvectors are inserted in column format.

$$Q = \left[\ \bar{q}_1 \mid \bar{q}_2 \ \right] \tag{38}$$

$$Q = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \tag{39}$$

The eigenvectors represent orthogonal mode shapes.

Each eigenvector can be multiplied by an arbitrary scale factor. A mass-normalized eigenvector matrix $\hat{Q}$ can be obtained such that the following orthogonality relations are obtained.

$$\hat{Q}^T \; M \; \hat{Q} = I \tag{40}$$

$$\hat{Q}^T \; K \; \hat{Q} = \Omega \tag{41}$$

where

superscript T represents transpose

I is the identity matrix

Ω is a diagonal matrix of eigenvalues

Note that

$$Q = \begin{bmatrix} \hat{v}_1 & \hat{w}_1 \\ \hat{v}_2 & \hat{w}_2 \end{bmatrix} \tag{42}$$

$$Q^T = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix} \tag{43}$$

Rigorous proof of the orthogonality relationships is beyond the scope of this tutorial. Further discussion is given in References 1 and 2.

Now define a modal coordinate $\eta(t)$ such that

$$\bar{x} = \hat{Q}\,\bar{\eta} \tag{44}$$

Substitute equation (44) into equation (9).

$$M\hat{Q}\,\ddot{\bar{\eta}} + C\hat{Q}\,\dot{\bar{\eta}} + K\hat{Q}\,\bar{\eta} = F \tag{45}$$

Premultiply by the transpose of the normalized eigenvector matrix.

$$\hat{Q}^T M\hat{Q}\,\ddot{\bar{\eta}} + \hat{Q}^T C\hat{Q}\,\dot{\bar{\eta}} + \hat{Q}^T K\hat{Q}\,\bar{\eta} = \hat{Q}^T F \tag{46}$$

The orthogonality relationships yield

$$I\,\ddot{\bar{\eta}} + \hat{Q}^T C \hat{Q}\,\dot{\bar{\eta}} + \Omega\,\bar{\eta} = \hat{Q}^T F \tag{47}$$

Furthermore, the following assumption is made.

$$\hat{Q}^T C \hat{Q}\,\dot{\bar{\eta}} = \begin{bmatrix} 2\xi_1\omega_1 & 0 \\ 0 & 2\xi_2\omega_2 \end{bmatrix} \tag{48}$$

where ξ_i is the modal damping ratio for mode i.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 2\xi_1\omega_1 & 0 \\ 0 & 2\xi_2\omega_2 \end{bmatrix}\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{49}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 2\xi_1\omega_1 & 0 \\ 0 & 2\xi_2\omega_2 \end{bmatrix}\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{50}$$

The two equations are now decoupled in terms of the modal coordinate.

$$\ddot{\eta}_1 + 2\xi_1\omega_1\,\dot{\eta}_1 + \omega_1^2\,\eta_1 = 0 \tag{51}$$

$$\ddot{\eta}_2 + 2\xi_2\omega_2\,\dot{\eta}_2 + \omega_2^2\,\eta_2 = 0 \tag{52}$$

Take the Laplace transform.

$$L\left\{\ddot{\eta}_1 + 2\xi_1\omega_1\,\dot{\eta}_1 + \omega_1^2\,\eta_1\right\} = 0 \tag{53}$$

$$s^2 \hat{\eta}_1(s) - s\eta_1(0) - \dot{\eta}_1(0) \quad + 2\xi_1\omega_1 s\, \hat{\eta}_1(s) - 2\xi_1\omega_1\eta_1(0) \quad + \omega_1^2\, \hat{\eta}_1(s) = 0 \tag{54}$$

$$\left\{ s^2 + 2\xi_1\omega_1 s + \omega_1^2 \right\} \hat{\eta}(s) - \dot{\eta}(0) - \left\{ s + 2\xi_1\omega_1 \right\} \eta(0) = 0 \tag{55}$$

$$\left\{ s^2 + 2\xi_1\omega_1 s + \omega_1^2 \right\} \hat{\eta}(s) = \dot{\eta}(0) + \left\{ s + 2\xi_1\omega_1 \right\} \eta(0) \tag{56}$$

$$\hat{\eta}(s) = \frac{1}{\left\{ s^2 + 2\xi_1\omega_1 s + \omega_1^2 \right\}} \dot{\eta}(0) + \frac{\left\{ s + 2\xi_1\omega_1 \right\}}{\left\{ s^2 + 2\xi_1\omega_1 s + \omega_1^2 \right\}} \eta(0) \tag{57}$$

Consider the denominator.

$$s^2 + 2\xi_1\omega_1 s + \omega_1^2 = (s + \xi_1\omega_1)^2 + \left(1 - \xi_1^2\right)\omega_1^2 \tag{58}$$

$$s^2 + 2\xi_1\omega_1 s + \omega_1^2 = (s + \xi_1\omega_1)^2 + \omega_{d1}^2 \tag{59}$$

Let

$$\omega_{d1} = \sqrt{1 - \xi_1^2}\ \omega_1 \tag{60}$$

By substitution,

$$\hat{\eta}(s) = \frac{1}{\left\{ (s + \xi_1\omega_1)^2 + \omega_{d1}^2 \right\}} \dot{\eta}(0) + \frac{\left\{ s + 2\xi_1\omega_1 \right\}}{\left\{ (s + \xi_1\omega_1)^2 + \omega_{d1}^2 \right\}} \eta(0) \tag{61}$$

$$\hat{\eta}(s) = \left\{\frac{1}{(s+\xi_1\omega_1)^2 + \omega_{d1}^2}\right\}\dot{\eta}(0) + \left\{\frac{\{s+2\xi_1\omega_1\}}{(s+\xi_1\omega_1)^2 + \omega_{d1}^2}\right\}\eta(0) \tag{62}$$

Take the Inverse Laplace transform.

$$\eta_1(t) = \exp(-\xi_1\,\omega_1\,t)\left\{\eta_1(0)\cos(\omega_{d1}\,t) + \frac{1}{\omega_{d1}}\left[\xi_1\omega_1\eta_1(0) + \dot{\eta}_1(0)\right]\sin(\omega_{d1}\,t)\right\}$$

(63)

$$\dot{\eta}_1(t) = -\xi_1\,\omega_1\exp(-\xi_1\,\omega_1\,t)\left\{\eta_1(0)\cos(\omega_{d1}\,t) + \frac{1}{\omega_{d1}}\left[\xi_1\omega_1\eta_1(0) + \dot{\eta}_1(0)\right]\sin(\omega_{d1}\,t)\right\}$$

$$+\exp(-\xi_1\,\omega_1\,t)\left\{-\omega_{d1}\eta_1(0)\sin(\omega_{d1}\,t) + \left[\xi_1\omega_1\eta_1(0) + \dot{\eta}_1(0)\right]\cos(\omega_{d1}\,t)\right\}$$

(64)

$$\dot{\eta}_1(t) = \exp(-\xi_1\,\omega_1\,t)\left\{-\omega_{d1}\eta_1(0) + \frac{-\xi_1\,\omega_1}{\omega_{d1}}\left[\xi_1\omega_1\eta_1(0) + \dot{\eta}_1(0)\right]\right\}\sin(\omega_{d1}\,t)$$

$$+\exp(-\xi_1\,\omega_1\,t)\left\{\dot{\eta}_1(0)\cos(\omega_{d1}\,t)\right\}$$

(65)

$$\dot{\eta}_1(t) = \exp(-\xi_1\,\omega_1\,t)\left\{\left\{-\omega_{d1} + \frac{-\xi_1^2\,\omega_1^2}{\omega_{d1}}\right\}\eta_1(0) + \frac{-\xi_1\,\omega_1\dot{\eta}_1(0)}{\omega_{d1}}\right\}\sin(\omega_{d1}\,t)$$

$$+\exp(-\xi_1\,\omega_1\,t)\left\{\dot{\eta}_1(0)\cos(\omega_{d1}\,t)\right\}$$

(66)

Similarly,

$$\eta_2(t) = \exp\left(-\xi_2\,\omega_2\,t\right)\left\{\eta_2(0)\cos\left(\omega_{d2}\,t\right) + \frac{1}{\omega_{d2}}\left[\xi_2\omega_2\eta_2(0) + \dot{\eta}_2(0)\right]\sin\left(\omega_{d2}\,t\right)\right\}$$

(67)

$$\bar{x} = \hat{Q}\,\bar{\eta} \qquad (68)$$

The displacements are

$$x_1(t) = v_1\,\eta_1(t) + w_1\,\eta_{2(t)} \qquad (69)$$

$$x_2(t) = v_2\,\eta_{1(t)} + w_2\,\eta_2(t) \qquad (70)$$

Now consider the initial conditions. Recall

$$\bar{x} = \hat{Q}\,\bar{\eta} \qquad (71)$$

Thus

$$\bar{x}(0) = \hat{Q}\,\bar{\eta}(0) \qquad (72)$$

Premultiply by $\hat{Q}^T M$.

$$\hat{Q}^T M\,\bar{x}(0) = \hat{Q}^T M\,\hat{Q}\;\bar{\eta}(0) \qquad (73)$$

Recall

$$\hat{Q}^T M\,\hat{Q} = I \qquad (74)$$

$$\hat{Q}^T M\,\bar{x}(0) = I\;\bar{\eta}(0) \qquad (75)$$

$$\hat{Q}^T M\,\bar{x}(0) = \bar{\eta}(0) \qquad (76)$$

Finally, the transformed initial displacement matrix is

$$\bar{\eta}(0) = \hat{Q}^{T} M \, \bar{x}(0) \tag{77}$$

Similarly, the transformed initial velocity is

$$\bar{\dot{\eta}}(0) = \hat{Q}^{T} M \, \bar{\dot{x}}(0) \tag{78}$$

A basis for a solution is thus derived.

<u>References</u>

1. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, New Jersey, 1982. Section 12.3.1.

2. Weaver and Johnston, Structural Dynamics by Finite Elements, Prentice-Hall, New Jersey, 1987. Chapter 4.

3. T. Irvine, Response of a Single-degree-of-freedom System Subjected to a Classical Pulse Base Excitation, Vibrationdata Publications, 1999.