Small Aircraft Gust Fatigue Spectra

Notes on the Use of Frequency Power-Spectrum Methods for Gust Fatigue Spectra Development on Rigid Airplanes

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Abstract. This paper describes an analytic method for the determination of gust load fatigue spectra considering flight condition and relevant aircraft aerodynamic/inertia characteristics. It applies to airplanes which are sufficiently rigid that during flight through turbulence the loads due to structural dynamic response are small in comparison to those arising from rigid-body motion. This is a condition met by the majority of small aircraft certified under CFR14 Part 23, and may be more generally applicable to certain other classes. No novel concepts are introduced. Rather, the method is the one already described for transport-category airplanes in CFR14 Part 25 Appendix G, though it is considered here only for the special case of rigid-body response. The aircraft frequency response function may then be defined analytically in terms of basic aerodynamic and inertia properties, and the resultant simplified computational procedures are easily implemented in an electronic spread-sheet, or by hand calculation.

Nomenclature.

$\bar{A}$ = root mean square response per unit gust
b = reference (wing) span, ft
b_1, b_2 = gust intensity coefficients (FAR25, App G)
c = reference (wing) chord, ft
C_X = aero axial force coefficient, F_X/QS, aft
C_Y = aero side force coefficient, F_Y/QS, right
C_Z = aero vertical force coefficient, F_Z/QS, up
C_l = aero rolling moment coefficient, M_X/QSb
C_m = aero pitching moment coefficient, M_Y/QSc
C_n = aero yawing moment coefficient, M_Z/QSb
f = frequency, Hz
F_n = force in direction n, lb
g = acceleration due to gravity =32.2 ft/sec^2
H = load factor transfer function (complex)
$|H|$ = amplitude of response to harmonic forcing $=\sqrt{(H_r^2+H_i^2)}$
I_{AA} = mass moment of inertia about axis A, sl-ft^2
I_{AB} = cross product of inertia about axes A,B
L = scale of turbulence (2,500 ft)
M = mass, slugs (= weight, lb / 32.2)
M_n = moment about axis n, ft-lb
N_Y = incremental lateral gust load factor, positive right
N_Z = incremental vertical gust load factor, positive up
N_0 = characteristic frequency, rad/ft
N(y)= number of positive or negative exceedances of value y in a given time or distance.
p = roll rate, positive right wing down, r/sec
p_1, p_2 = gust probability coefficients
q = pitch rate, nose up, r/sec
Q = dynamic pressure, lb/ft^2
r = yaw rate, r/sec, positive aircraft nose right
rms = "root-mean-square"
Re = real part of a complex value.
S = reference area, ft^2
u_g = gust velocity, ft/sec true
u_{de} = root-mean-square gust intensity, ft/sec true
V = aircraft true velocity, ft/sec
W = airplane weight, lb
α = angle of attack, rad
β = sideslip, positive aircraft nose left
ϕ = bank angle, positive right wing down, rad
θ = pitch attitude, nose up, rad
Φ_g = gust PSD, ft^3/(rad.sec^2)
Φ_{Nz} = load factor PSD, ft/rad
μ = airplane mass ratio, M/QS
ω = frequency, rad/sec
ω_0 = natural frequency of unforced motion, rad/sec
Ω= reduced frequency, rad/ft
ζ = damping ratio of unforced motion

Background. Substantial amounts of operational data have been gathered for load exceedance counts during flight through turbulence, and these are available for predictive analysis of likely exceedance spectra on new small airplane programs. References 5,6,7,8, and 9 are some of the data sources which have been used to develop design data. However, it is not necessarily the case that measured data from previous designs can be applied to a new design. An inherent difficulty is that gust response is strongly influenced by aircraft configuration, gross weight at the time of the event, airspeed, and altitude, and this combination of factors can cause a marked variability in the severity of repeated gust loads between aircraft models and operating conditions. The problem is exacerbated by recent developments which have seen the flight envelope of small aircraft expand into areas once reserved for larger aircraft. Thus the full range of small-aircraft operations now ranges between "low-and-slow" trainer aircraft with modest wing loading, to high-altitude operations of business jets with swept, highly-loaded wings and transonic capability. Other special-use categories such as agricultural or fire-fighting operations further expand the range of missions which might need to be addressed. Each of these types operate in the same atmosphere, and yet have significantly different gust load spectra.

With this in mind, a rigorous method is sought which is able to calculate airplane gust fatigue spectra considering all relevant airplane characteristics and operating envelope variables, and therefore able to predict appropriate spectra for each fatigue mission flight segment.

Mission Analysis Overview.

Mission analysis is a well-established power spectrum method implemented for large aircraft design in Reference (4), Section (c). An excellent discussion of the subject will be found in Reference 1. Since a complete knowledge of the mathematical basis is not needed for present purposes, only a summary is presented here.

Briefly, the method is based on assumptions that continuous random turbulence can be idealized such that (i) turbulence is isotropic, with an equal probability of gust velocity in any direction, (ii) random measurement of gust velocity would follow a Gaussian distribution, and (iii) the frequency content of turbulence is unique, and invariant over time. This latter amounts to the specification of a single power spectral density function (PSD) of atmospheric turbulence intensity. The standard turbulence model used here is the so-called Von Karman gust PSD function defined in Reference (4).

With these assumptions, random turbulence can be synthesized from an infinite number of randomly-phased elemental harmonic gusts occupying the complete spectrum from zero frequency to infinite frequency. The PSD function defines the relative strength of these elemental harmonic gusts as a function of frequency, where the term "strength" is interpreted as their contribution to the overall mean-square gust velocity. Figure 1 and Equation (1) define the Von Karman spectrum for a gust field of unit root-mean-square gust intensity.

$$\Phi_g(\Omega) = \frac{L}{\pi}\frac{1+8/3(1.339L\Omega)^2}{[1+(1.339L\Omega)^2]^{11/6}} \qquad (1)$$

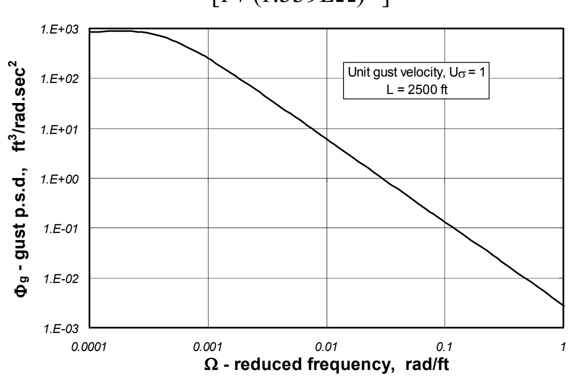


Fig 1: *Von Karman Gust Spectrum*

Note that the reduced frequency Ω is a spatial measurement of the frequency at which gusts are encountered in terms of radians per unit distance traveled along the flight path, and is related to the temporal frequency ω at which the airplane encounters gust peaks through the true airspeed V:

$$\omega\,\frac{rad}{\sec} = \Omega V\,\frac{rad}{ft}\frac{ft}{\sec}$$

Consider a continuous harmonic gust (Fig 4) and let H(Ω) be the transfer function which defines the response of some aircraft response parameter [N_Z, say] to unit amplitude gust forcing at frequency Ω, so that:

$$|H_{Nz}(\Omega)| = |N_Z|/|u_g| \qquad g\,/\,(\text{ft/sec})$$

then the PSD of load factor response N_Z will be given by:

$$\Phi_{Nz} = |H_{Nz}|^2\,\Phi_g \qquad g^2\,/\,(\text{rad/ft}) \qquad (2)$$

Any response parameter or load could be considered, but here we will consider only two, (a) the incremental vertical load factor N_Z at the aircraft center of gravity for symmetric vertical gusts, or (b) the incremental lateral load factor N_Y at the aircraft center of gravity for lateral gusts.

Then, for the complete spectrum of infinitesimal gusts having unit total rms gust intensity (u_{de} = 1) distributed per the Von Karman spectrum, the rms N_Z response is given by:

$$\overline{A}_{Nz} = \sqrt{\int_0^\infty \Phi_{Nz}\,d\Omega} \qquad g's\,/\text{ft/sec} \qquad (3)$$

Also, the characteristic frequency of N_Z is given by:

$$N_{0,Nz} = \frac{1}{\overline{A}_{Nz}}\sqrt{\int_0^\infty \Phi_{Nz}\Omega^2\,d\Omega} \quad \text{rad / ft} \qquad (4)$$

The usual approach is to discretize the frequency spectrum and determine the value of *H(Ω)* at a large number of fixed forcing frequencies *Ω*, and to replace the integrals of Equations (3) and (4) with equivalent summations over finite intervals.

$$\overline{A}_{Nz} = \sqrt{\sum\nolimits_n \Phi_{Nz}\Delta\Omega} \qquad g\,/\,(\text{ft/sec}) \qquad (5)$$

$$N_{0,Nz} = \frac{1}{\overline{A}_{Nz}}\sqrt{\sum\nolimits_n \Phi_{Nz}\Omega^2\Delta\Omega} \qquad \text{rad/ft} \qquad (6)$$

There is no difficulty with the infinite upper limit of integration in (3) since it will be found that $\bar{A}$ tends to an asymptotic limit, and the integration can be truncated without loss of accuracy at an appropriate upper cut-off frequency. The integral defined for N_0 in (4) does not generally close to an asymptotic value at high frequency, and will usually be obtained by integrating only up to the same upper frequency cut-off as defined for $\bar{A}$ (see Reference 1, Appendix E, for discussion on cut-off frequency).

Characteristic frequency $N_{0,Nz}$ provides a measure of the frequency of occurrence of positive N_Z response peaks, representing the average time or distance traveled between successive positive crossings of N_Z=0 (see figure 2). Between each such crossing there is one positive maximum N_Z value. At airspeed V the *average* number of positive (or negative) peaks per unit time is given by:

$$\text{Average positive } N_Z \text{ counts} = \frac{N_{0,Nz}V}{2\pi} \text{ per second}$$

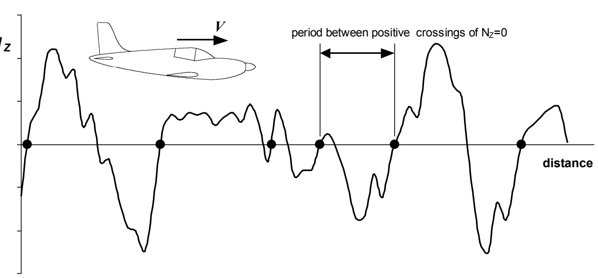


Fig 2 *Gust Load Factor Time History*

The airplane is not operated exclusively in turbulence. Most of the time it will be flown in a smooth atmosphere, and for those times when it is subject to turbulence the gust field is liable to be of variable intensity. Studies of atmospheric gust characteristics have shown that turbulence encounters can be roughly split into two categories; relatively frequent low-intensity gust fields ("non-storm turbulence") and less frequent high-intensity gust fields ("storm turbulence"). The statistical probability of being exposed to storm and non-storm turbulence, and the severity of gust intensity during exposure, has been estimated and this information is captured in the so-called "*p*'s and *b*'s". These are properties of the atmosphere only, unrelated to aircraft type or operating condition.

As a result of the formulations employed a fundamental relationship is obtained for $N(N_Z)$, the average number of peaks exceeding a given value of N_Z in a flight segment of duration T seconds, considering the combined probability of exposure to smooth air, non-storm turbulence, and storm turbulence. This relationship is:

$$N(N_Z) = T\frac{N_{0,Nz}V}{2\pi}\left\{p_1 e^{-Nz/b_1\bar{A}} + p_2 e^{-Nz/b_2\bar{A}}\right\} \qquad (7)$$

A typical exceedance spectrum is shown in Figure 3.

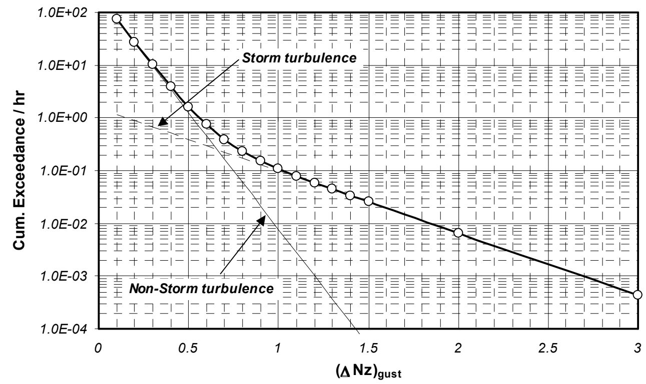


Fig 3 *Typical Exceedance Curve*

Equation (7) is the desired incremental load factor exceedance spectrum at a specific flight envelope point In this equation the *p*'s represent the fraction of flight time the aircraft is exposed to non-storm and storm turbulence. [p_1 = non-storm, p_2 = storm turbulence, while the probability of being in smooth air is $(1- p_1 - p_2)$]. The *b*'s may be thought of as scalars which represent the gust intensity occurring in non-storm and storm turbulence. Appropriate values are defined as a function of altitude above sea level in reference (4).

The utility of the power-spectrum method is therefore that spectrum definition is reduced to only the determination the transfer function $|H_{Nz}|$ for harmonic forcing. This, together with the Von Karman unit rms gust spectrum, leads directly to the Nz response spectrum using equation (2) and then to the parameters $\bar{A}_{Nz}$ and $N_{0,Nz}$ using equations (5) and (6).

In the present case we have a system with a limited number of rigid-body degrees of freedom, and an opportunity to establish an analytic definition of the load factor transfer functions.

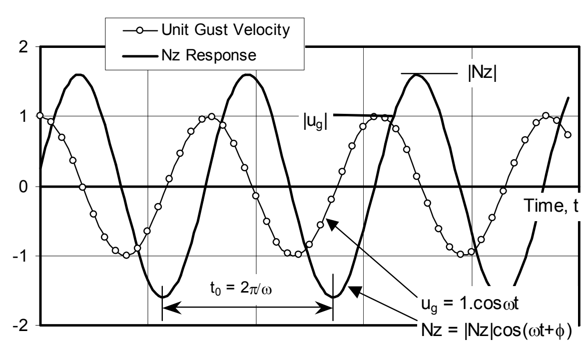


Fig 4 *Steady State Response to Unit Continuous Harmonic Gust*

The nature of steady-state incremental load factor response is illustrated in Figure 4. At any fixed forcing frequency a unit harmonic gust, u_g=1, will drive N_Z at the same frequency, but there will be both a phase-shift and gain between u_g (input) and N_Z (output).

Rigid-Body Gust Response

Equations of Motion. The exact six degree-of-freedom equations of motion, when linearized for small perturbations about initial datum conditions (denoted by subscript 0), become de-coupled and may be separated into symmetric and asymmetric 3-freedom sets:

Symmetric equations of motion

$$
\begin{aligned}
C_X + \mu g\theta\cos\theta_0 &= -\mu\left(\dot{V}\cos\alpha_0 + V_0\sin\alpha_0(q-\dot{\alpha})\right)\\
C_Z + \mu g\theta\sin\theta_0 &= -\mu\left(\dot{V}\sin\alpha_0 + V_0\cos\alpha_0(\dot{\alpha}-q)\right)\\
C_m &= I_{YY}\dot{q}/QSc
\end{aligned}
\tag{8a}
$$

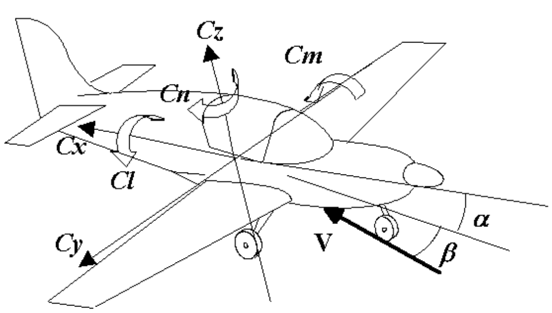
Lateral / directional equations of motion

$$
\begin{aligned}
C_Y + \mu g\phi\cos\theta_0 &= \mu V_0\left(\dot{\beta} + r\cos\alpha_0 - p\sin\alpha_0\right)\\
C_l &= \left(I_{XX}\dot{p} - I_{XZ}\dot{r}\right)/QSb\\
C_n &= \left(I_{ZZ}\dot{r} - I_{XZ}\dot{p}\right)/QSb
\end{aligned}
\tag{8b}
$$

Neglecting unsteady terms the airplane aerodynamic force coefficients may be written as the sum of linear derivatives, as follows:

$$
\begin{aligned}
C_X &= C_{X\alpha}\alpha + C_{Xq}q + C_{XV}V + (C_X)_{gust}\\
C_Z &= C_{Z\alpha}\alpha + C_{Zq}q + C_{ZV}V + (C_Z)_{gust}\\
C_m &= C_{m\alpha}\alpha + C_{mq}q + C_{mV}V + (C_m)_{gust}
\end{aligned}
\tag{9a}
$$

$$
\begin{aligned}
C_Y &= C_{Y\beta}\beta + C_{Yp}p + C_{Yr}r + (C_Y)_{gust}\\
C_l &= C_{l\beta}\beta + C_{lp}p + C_{lr}r + (C_l)_{gust}\\
C_n &= C_{n\beta}\beta + C_{np}p + C_{nr}r + (C_n)_{gust}
\end{aligned}
\tag{9b}
$$



(Note that for small angles
$C_Z = C_L\cos\alpha + C_D\sin\alpha \approx C_L$, where C_L = lift coefficient.,
$C_X = C_D\cos\alpha - C_L\sin\alpha \approx C_D$, where C_D = drag coeff.)

A vertical gust of modulus $|u_g|$ imposes a continuous harmonic angle of attack = $(u_g/V)\cos\omega t$, while a lateral gust of modulus $|u_g|$ imposes a continuous harmonic sideslip = $(u_g/V)\cos\omega t$. Hence vertical and lateral gust forces are represented by, for example:

$$(C_Z)_{gust} = C_{Z\alpha}(u_g/V)\cos\omega t \tag{10a}$$

$$(C_y)_{gust} = C_{y\beta}(u_g/V)\cos\omega t \tag{10b}$$

<u>Equations of Symmetric Motion</u> : Combining (8a), (9a), (10a), and noting that $q = d\theta/dt$ the symmetric equations of motion may be written:

$$[A]_{4,4}\{x\}_4 + [B]_{4,4}\{\dot{x}\}_4 = \{F_{gust}\}_4 \tag{11}$$

where:

$$\{x\} = \begin{Bmatrix}\alpha\\ \theta\\ q\\ V\end{Bmatrix}$$

$$[A] = \begin{bmatrix} C_{X\alpha} & \mu g\cos\theta_0 & C_{Xq} & C_{XV}\\ C_{Z\alpha} & \mu g\sin\theta_0 & C_{Zq} & C_{ZV}\\ C_{m\alpha} & 0 & C_{mq} & C_{mV}\\ 0 & 0 & 1 & 0\end{bmatrix}$$

$$[B] = \begin{bmatrix} -\mu V_0\sin\alpha_0 & \mu V_0\sin\alpha_0 & 0 & \mu\cos\alpha_0\\ \mu V_0\cos\alpha_0 & -\mu V_0\cos\alpha_0 & 0 & \mu\sin\alpha_0\\ 0 & 0 & -I_{YY}/QSc & 0\\ 0 & -1 & 0 & 0\end{bmatrix}$$

$$\{F_{gust}\} = -(u_g/V)\begin{Bmatrix}C_{X\alpha}\\ C_{Z\alpha}\\ C_{m\alpha}\\ 0\end{Bmatrix}\cos\omega t$$

<u>Equations of Lateral/Directional Motion</u> : Combining (8b), (9b), (10b), and noting that $p = d\phi/dt$ the asymmetric equations of motion may be written:

$$[A]_{4,4}\{x\}_4 + [B]_{4,4}\{\dot{x}\}_4 = \{F_{gust}\}_4 \tag{12}$$

where:

$$\{x\} = \begin{Bmatrix}\beta\\ \phi\\ p\\ r\end{Bmatrix}$$

$$[A] = \begin{bmatrix} C_{Y\beta} & \mu g\cos\theta_0 & C_{Yp} + \mu V_0\sin\alpha_0 & C_{Yr} - \mu V_0\cos\alpha_0\\ C_{l\beta} & 0 & C_{lp} & C_{lr}\\ C_{n\beta} & 0 & C_{np} & C_{nr}\\ 0 & 0 & 1 & 0\end{bmatrix}$$

$$[B]=\begin{bmatrix}-\mu V_0 & 0 & 0 & 0\\ 0 & 0 & -I_{XX}/QSb & I_{XZ}/QSb\\ 0 & 0 & I_{XZ}/QSb & -I_{ZZ}/QSb\\ 0 & -1 & 0 & 0\end{bmatrix}$$

$$\{F_{gust}\}=-(u_g/V)\begin{Bmatrix}C_{Y\beta}\\ C_{l\beta}\\ C_{n\beta}\\ 0\end{Bmatrix}\cos\omega t$$

Numerical Solution

The equations of vertical and lateral gust response, (11) and (12), are eigenvalue problems, and in each case it is anticipated that the steady-state solution for $\{x\}$ will be harmonic, at the same frequency ω as gust forcing, so we start by assuming that the solution is as follows:

$$\{x\} = \{X\}\cos\omega t = Re\left[\{X\}e^{i\omega t}\right]$$

whereupon $\{\dot{x}\}=\mathrm{Re}\left[i\omega\{X\}e^{i\omega t}\right]$

Substituting into either (11) or (12) and dividing by the common factor $e^{i\omega t}$, we obtain:

$$[[A]+i\omega[B]]\{X\}=\{F\}$$

or :

$$\{X\}=\left[C^{-1}\right]\{F\} \qquad where \quad [C]=[A]+i\omega[B]$$

The solution is therefore:

$$\{x\}=\mathrm{Re}\left[\{X\}e^{i\omega t}\right]=\mathrm{Re}\left[\left[C^{-1}\right]\{F\}e^{i\omega t}\right] \tag{13}$$

Both the matrix [C] and its inverse $[C^{-1}]$ are complex, but if we separate $[C^{-1}]$ into real and imaginary components such that:

$$[C^{-1}]\equiv[D]+i[E]$$

where [D] and [E] are real, we obtain:

$$\begin{aligned}\{x\}&=\mathrm{Re}\left[([D]+i[E])\{F\}(\cos\omega t+i\sin\omega t)\right]\\ &=([D]\cos\omega t-[E]\sin\omega t)\{F\}\end{aligned} \tag{14}$$

Provided the relevant inertia and aerodynamic coefficient data is known, Equation (14) provides an exact solution for the steady-state response of the aircraft to gust forcing at any defined forcing frequency ω. It will be noted that ω is embedded in matrix [C], so a new matrix inversion is required for each new frequency.

The normal load factor response of the aircraft is obtained as follows. (lateral load factor response is not shown but is analogous).

$N_Z(t)= F_Z(t)/W = C_Z(t)QS/W$

where F_Z is the total aerodynamic vertical force, this being the linear superposition of the forces due to the gust, $u_g(t)$, plus the forces due to aircraft response, $\{x(t)\}$. From (9a) and (14) the vertical force due to aircraft motion is:

$$\begin{aligned}(F_z)_{motion}&=QS\{C_{Z\alpha},C_{Zq},C_{ZV},0\}_{1,4}\{x\}_{4,1}\\ &\equiv QS[M]_{1,4}\{x\}_{4,1}\\ &=QS[M][D]\{F\}\cos\omega t-QS[M][E]\{F\}\sin\omega t\\ &=QS(k_1\cos\omega t+k_2\sin\omega t)\end{aligned}$$

While the vertical force due to gust is:

$$(F_Z)_{gust}=QSC_{Z\alpha}(u_g/V)\cos\omega t$$

The vertical force due to aircraft response therefore has a component in phase with the gust, and a component 90° out of phase, and the amplitude of load factor response is given by:

$$|N_Z|=\frac{QS}{W}\sqrt{\left(C_{Z\alpha}\frac{u_g}{V}+k_1\right)^2+k_2{}^2} \tag{15}$$

Analytic Solution

The solution outlined above is by way of numerical matrix inversion, once for each frequency, and while there is no difficulty in implementing this in an Excel spreadsheet it would require repeated execution over a range of forcing frequencies in order to fully define the transfer function. An analytic solution which is valid for any arbitrary forcing frequency provides an easier route to an automated procedure for fatigue spectrum development. Such solutions are available using Laplace transform methods.

Vertical Gust Response : the symmetric equations of motion, 8(a), are reduced to two degrees-of-freedom in pitch and vertical heave, by neglecting the axial force balance on the assumption of constant airspeed. The resultant 2 d-o-f system has a single complex-conjugate root corresponding to short-period mode. Also, the datum pitch attitude and angle of attack may be assumed small, such that $\cos\alpha_0 = \cos\theta_0 = 1$; $\sin\alpha_0 = \sin\theta_0 = 0$. The small perturbation vertical force and pitching moment equations 8(a) are then simplified to:

$$\begin{aligned}&F_Z+MV(\dot{\alpha}-q)=0\\ &M_Y-Iyy\dot{q}=0\end{aligned} \tag{16}$$

Including the effect of a continuous harmonic gust $u_g(t) = u_g \sin\omega t$, the aerodynamic forces are defined by:

$$\begin{aligned} F_Z &= QS\left(C_{Z\alpha}\alpha + C_{Zq}q + C_{Z\alpha}\left(u_g / V\right)\sin\omega t\right) \\ M_Y &= QSc\left(C_{m\alpha}\alpha + C_{mq}q + C_{m\alpha}\left(u_g / V\right)\sin\omega t\right) \end{aligned} \tag{17}$$

Combining (16) and (17):

$$\begin{aligned} k_1\alpha + k_2\dot{\alpha} + k_3 q &= g_1 \sin\omega t \\ k_4\alpha + k_5 q + k_6\dot{q} &= g_2 \sin\omega t \end{aligned} \tag{18}$$

where

$$\begin{aligned} &k_1 = QSC_{Z\alpha} \qquad k_2 = MV \qquad k_3 = QSC_{Zq} - MV \\ &k_4 = QScC_{m\alpha} \qquad k_5 = QScC_{mq} \qquad k_6 = -I_{YY} \\ &g_1 = -QSC_{Z\alpha}(u_g / V) \qquad g_2 = -QScC_{m\alpha}(u_g / V) \end{aligned}$$

Take the Laplace transforms of equations (18) with initial conditions $\alpha(0)=q(0)=0$:

$$\begin{aligned} (k_1 + k_2 s)\overline{\alpha} + k_3\overline{q} &= \frac{g_1\omega}{s^2 + \omega^2} \\ k_4\overline{\alpha} + (k_5 + k_6 s)\overline{q} &= \frac{g_2\omega}{s^2 + \omega^2} \end{aligned} \tag{19}$$

and solve for $\overline{\alpha}$ by eliminating $\overline{q}$ from equations (19) to obtain:

$$\overline{\alpha} = \frac{f_1\omega + f_2\omega s}{\left(s^2 + \omega^2\right)\left(s^2 + d_2 s + d_1\right)} \tag{20}$$

where:

$$\begin{aligned} f_1 &= (k_5 g_1 - k_3 g_2) / k_2 k_6 \\ f_2 &= g_1 / k_2 \\ d_1 &= (k_1 k_5 - k_3 k_4) / k_2 k_6 \\ d_2 &= (k_2 k_5 + k_1 k_6) / k_2 k_6 \end{aligned}$$

Note the following identity, which may be readily verified from the characteristic equation of unforced motion:

$$\begin{aligned} \left(s^2 + 2\zeta\omega_0 s + \omega_0{}^2\right) &\equiv \left(s^2 + d_2 s + d_1\right) \\ &= \left((s-a)^2 + b^2\right) \end{aligned}$$

where
ω_0 = natural frequency of unforced short-period mode
ζ = damping ratio

Hence:

$$\begin{aligned} &\omega_0 = \sqrt{d_1} \\ &\zeta = d_2 / 2\sqrt{d_1} \\ &a = -d_2 / 2 \quad = -\zeta\omega_0 \\ &b = \sqrt{4d_1 - d_2{}^2} / 2 \quad = \omega_0\sqrt{1-\zeta^2} \end{aligned}$$

Equation (20) has a partial-fraction expansion of the form:

$$\overline{\alpha} = \frac{A\omega}{(s^2+\omega^2)} + \frac{Bs}{(s^2+\omega^2)} + \frac{Cb}{(s-a)^2 + b^2} + \frac{D(s-a)}{(s-a)^2 + b^2} \tag{21}$$

where A,B,C and D are constants, and for which the solution in the time domain may be directly written by inspection as:

$$\alpha(t) = A\sin\omega t + B\cos\omega t + Ce^{at}\sin bt + De^{at}\cos bt \tag{22}$$

Comparing the numerators of (21) and (20), and equating the coefficients of like powers of *s* we obtain four linear equations in the coefficients A,B,C and D

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \omega & -2a & b & -a \\ -2a\omega & (a^2+b^2) & 0 & \omega^2 \\ \omega(a^2+b^2) & 0 & b\omega^2 & -a\omega^2 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f_2\omega \\ f_1\omega \end{Bmatrix}$$

the above equations can be solved numerically, although, after some algebra, explicit expressions for the coefficients can be derived as follows:

$$A = \frac{f_1(\omega_0{}^2 - \omega^2) + f_2\omega(2\zeta\omega\omega_0)}{[(\omega_0{}^2 - \omega^2)^2 + (2\zeta\omega\omega_0)^2]} \tag{23a}$$

$$B = \frac{-f_1(2\zeta\omega\omega_0) + f_2\omega(\omega_0{}^2 - \omega^2)}{[(\omega_0{}^2 - \omega^2)^2 + (2\zeta\omega\omega_0)^2]} \tag{23b}$$

$$C = \frac{1}{2\omega\omega_0\sqrt{1-\zeta^2}} \cdot \frac{-2f_1\omega^2\left(\omega_0{}^2 - \omega^2\right) + f_1\left(2\zeta\omega\omega_0\right)^2 - f_2\omega\left(2\zeta\omega\omega_0\right)\left(\omega^2 + \omega_0{}^2\right)}{\left(\omega_0{}^2 - \omega^2\right)^2 + \left(2\zeta\omega\omega_0\right)^2}$$

$$D = -B$$

Coefficients A and B determine the steady-state (general) solution, while coefficients C and D represent a transient (particular) solution for the specified boundary conditions $\alpha(0)=q(0)=0$. We shall require only A and B.

Normal load factor is assumed to be a function of α only, where α is now considered to be the total angle of attack arising from the incident gust plus resultant aircraft response, and making use of (22) we have:

$$\begin{aligned} \Delta N_Z(t) &= \frac{C_{Z\alpha}\alpha(t).QS}{W} \\ &= \frac{C_{Z\alpha}QS}{W}\left[\left(A\sin\omega t + B\cos\omega t\right) + (u_g / V)\sin\omega t\right] \end{aligned}$$

So the modulus of steady-state normal load factor response is given by:

$$\left|H_{Nz}\right| = \frac{|N_Z|}{|u_g|} = \frac{C_{Z\alpha}QS}{u_g W}\sqrt{\left(A + u_g / V\right)^2 + B^2} \tag{24}$$

Lateral Gust Response : For ease of analysis the lateral/direction equations of motion 8(b) are reduced to a two degree-of-freedom system in yaw and lateral heave only. Freedom in roll is neglected, so that $\phi = p = 0$. The resultant system has a single pair of complex conjugate roots with a natural frequency which approximates that of the dutch roll mode. The datum pitch attitude and angle of attack are assumed to be small, such that $\cos\alpha_0 = \cos\theta_0 = 1$; $\sin\alpha_0 = \sin\theta_0 = 0$. The small perturbation side-force and yawing moment equations 8(b) now become:

$$
\begin{aligned}
F_Y - MV\left(\dot{\beta} + r\right) = 0 \\
M_Z - I_{ZZ}\dot{r} = 0
\end{aligned} \tag{25}
$$

Including the effect of a continuous lateral harmonic gust $u_g(t) = u_g \sin\omega t$, the aerodynamic forces are defined by:

$$
\begin{aligned}
F_Y = QS\left(C_{Y\beta}\beta + C_{Yr}r + C_{Y\beta}\left(u_g / V\right)\sin\omega t\right) \\
M_Z = QSb\left(C_{n\beta}\beta + C_{nr}r + C_{n\beta}\left(u_g / V\right)\sin\omega t\right)
\end{aligned} \tag{26}
$$

Combining (25) and (26):

$$
\begin{aligned}
k_1\beta + k_2\dot{\beta} + k_3 r = g_1 \sin\omega t \\
k_4\beta + k_5 r + k_6\dot{r} = g_2 \sin\omega t
\end{aligned} \tag{27}
$$

where

$$
\begin{aligned}
&k_1 = QSC_{Y\beta} \qquad k_2 = -MV \qquad k_3 = QSC_{Yr} - MV \\
&k_4 = QSbC_{n\beta} \qquad k_5 = QSbC_{nr} \qquad k_6 = -I_{ZZ} \\
&g_1 = -QSC_{Y\beta}(u_g / V) \qquad g_2 = -QSbC_{n\beta}(u_g / V)
\end{aligned}
$$

Equation (27) will be recognized to be identical in form to (18). Hence the solutions for steady-state sideslip and lateral load factor response, $\beta(t)$ & N_Y, are identical in form to that shown for vertical gust response in (22) and (24), except that coefficient values A and B are now to be derived using the coefficient constants specified in this section.

Step-by-Step Calculation Procedure :

1. Based on aerodynamic coefficient data at the flight condition under consideration, and the inertia properties of the airplane, determine the coefficient values k_1 through k_6, g_1, g_2, per eqn (18). Note that aerodynamic coefficients are per radian and are at the airplane center of gravity. Use $u_g = 1$, unit gust amplitude.
2. Calculate f_1, f_2, d_1, d_2, and hence the unforced natural frequency ω_0 and damping ratio ζ, eqn (20).
3. Determine coefficients A and B, eqn (23), for a range of discrete forcing frequencies. Start around $\omega = 0.5$ rad/sec and continue to around 30 rad/sec in 0.5 rad/sec increments
4. Hence determine the value of the load factor transfer function $|H(\omega)|$ at each selected frequency, eqn (24). For each frequency ω determine reduced frequency Ω, and thereby define $|H(\Omega)|$.
5. Multiply the gust PSD values from the Von Karman spectrum, Eqn 1, with $|H(\Omega)|^2$ to obtain the load factor power spectrum Φ_{Nz}, Eqn. 2.
6. Numerically integrate the area under the load factor PSD curve, terminating the integration when the integral nears its asymptotic maximum, Eqn 5. Note the cut-off frequency. Determine $\bar{A}_{Nz}$ as the square root of the integral value.
7. Perform the similar integration and root-taking process defined in Eqn 6 to obtain $N_{0,Nz}$. Use the same cut-off frequency as in step 6.
8. Refer to Reference (4) to establish the numeric values of p_1, p_2, b_1, b_2 at the altitude under consideration.
9. Apply the derived data values in Eqn 7 to determine the fatigue load factor spectrum. Use T = 3600 seconds to obtain load factor exceedances per hour.

Results and Discussion

Techniques suitable for small aircraft design are often those requiring minimal aerodynamic design data, and it will be noted from (17) and (26) that only the following terms are specified as input:

(a) For vertical gust analysis: $C_{Z\alpha}$, $C_{m\alpha}$, C_{Zq}, C_{mq}.

(b) For lateral gust analysis: $C_{Y\beta}$, $C_{n\beta}$, C_{Yr}, C_{nr}

Figure 5 shows transfer functions H_{Nz} and H_{Ny} for different solution methods and data availability, using a specimen aircraft and flight condition, and demonstrates the following:

(i) A 2-dof pitch/heave vertical gust solution very closely matches the 3-dof numerical solution, proving the validity of the constant-speed assumption.

(ii) Pitch rate terms are significant in determining resonant response near short-period frequency. Omission of pitch damping terms C_{Zq} and C_{mq} causes locally increased amplification, and hence an over-estimate of the gust load spectrum severity.

(iii) A 3-dof numerical solution which is extended to include un-steady aerodynamic terms $C_{Z\dot{\alpha}}, C_{m\dot{\alpha}}, C_{Z\dot{q}}, C_{m\dot{q}}$ results in minimal change to the transfer function, verifying that unsteady aerodynamic effects are generally negligible.

(iv) Comparative aircraft response parameters at a high-speed condition (V_C at sea level) are as follows:

	$\bar{A}_{Nz}$ (g)	$N_{0,Nz}$ (Hz)
2-dof, analytic	0.0367	1.230
3-dof, exact numerical	0.0367	1.230
2-dof, no pitch damping	0.0465	1.064
3-dof + unsteady aero	0.0368	1.248

(v) A 2-dof yaw/heave lateral gust solution which omits freedom in roll is a good approximation to the exact 3-dof solution. Neglect of the roll freedom has little impact.

(vi) Omission of the yaw damping terms C_{Yr}, C_{nr} for lateral gust gives artificially increased aircraft response near dutch roll frequency, and results in a somewhat conservative estimate of lateral spectrum severity.

It can be concluded that the most accurate solutions require the inclusion of aerodynamic rate damping terms, even if these are only estimated values due to the empennage, which is usually the primary source. The use of only angle of attack (or sideslip) derivatives will result in slightly conservative estimates of fatigue spectra.

Table 1 and Figure 6 present representative mission-analysis fatigue spectra for different categories of small aircraft operating at nominal cruise conditions. It will be seen that the load factor frequency of exceedance is expected to vary widely across the range of models, by 2-3 orders of magnitude.

Figures 7,8,9 and 10(a) show comparisons of mission-analysis with results from AC23-13A and ESDU 69023. In general, there is reasonable agreement between all data sources at nominal cruise conditions, with the exception of aircraft operating at the higher altitude typical for a light business jet (above 20,000 ft). The diminished probability of turbulence with increasing altitude is accounted for by ESDU 69023 and the present method, but not by AC23-13A (see Fig 7(a)). For this same business jet operating at a lower altitude and speed the agreement between all methods is good (see Fig 7(b)).

AC23-13A differentiates between selected aircraft types and missions by providing a different spectrum for each, for example agricultural crop-duster usage shown in Fig 8. The present mission-analysis technique is a single method which appears able to match these various spectra by the use of appropriate basic airplane data and operating condition, and has the further advantage that it will automatically accommodate any defined variations from the mean, as may occur within an overall aircraft category.

Figures 10(b),(c) and (d) illustrate fatigue spectrum variability for one of the base-lined models, a pressurized twin-turboprop. Airspeed, altitude, weight and CG position are all seen to have substantial effects on the repeated gust loading, with specific consequences for individual designs. Not shown are the effects of airplane configuration on aerodynamic characteristics, which may give rise to additional variability. STOL aircraft with large tail volume, or specialist aircraft with ultra-high aspect ratio wings, are among those where wing loading and lift curve slope may alone not be sufficient to fully characterize gust response. The present method may therefore be of particular value when applied to specialist configurations, unusual operating conditions, or in determining the incremental effects of changing operating conditions on a given airplane model, as for example in adapting an aircraft to a new low-altitude mission.

The present method relies on FAR 25 Appendix G for the atmospheric turbulence definition provided by the p's and b's. While these data have been used successfully since their introduction in the 1960's it should be noted that gust properties are to some extent dependant on the operating limitations of the aircraft used in their measurement. Thus an aircraft which today has weather radar will often divert around storms and not be exposed to the worst atmospheric conditions. To the extent that current General Aviation airplanes operate in different circumstances than air transport operations in the 1960's some future re-appraisal of the gust coefficient data for small aircraft might be in order.

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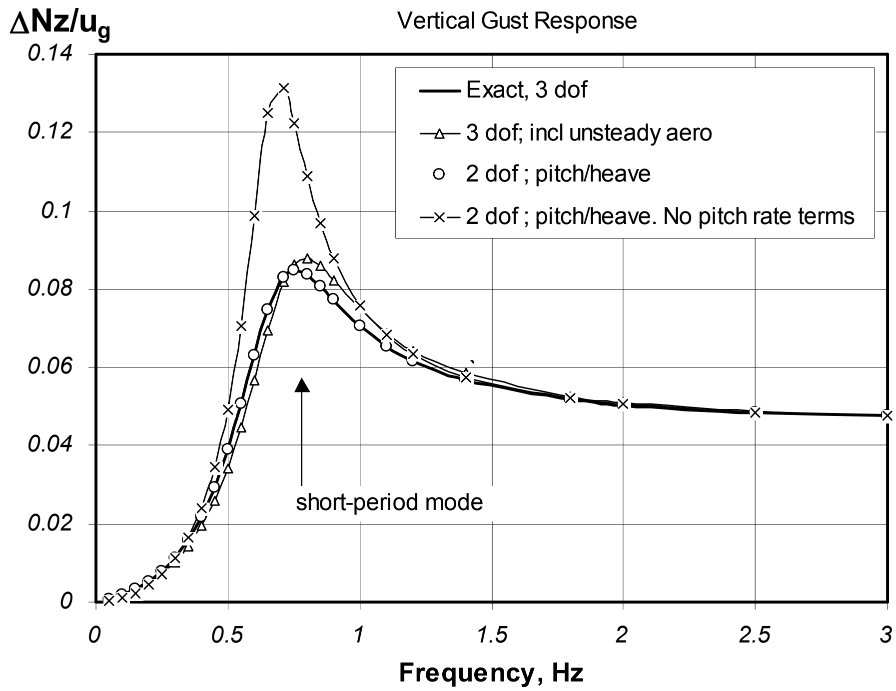
Figure 5 C.G. Load Factor Transfer Functions, $|H_{Nz}|$ and $|H_{Ny}|$
For Various Levels of Solution
(Light Business Jet at V_C/sea level)



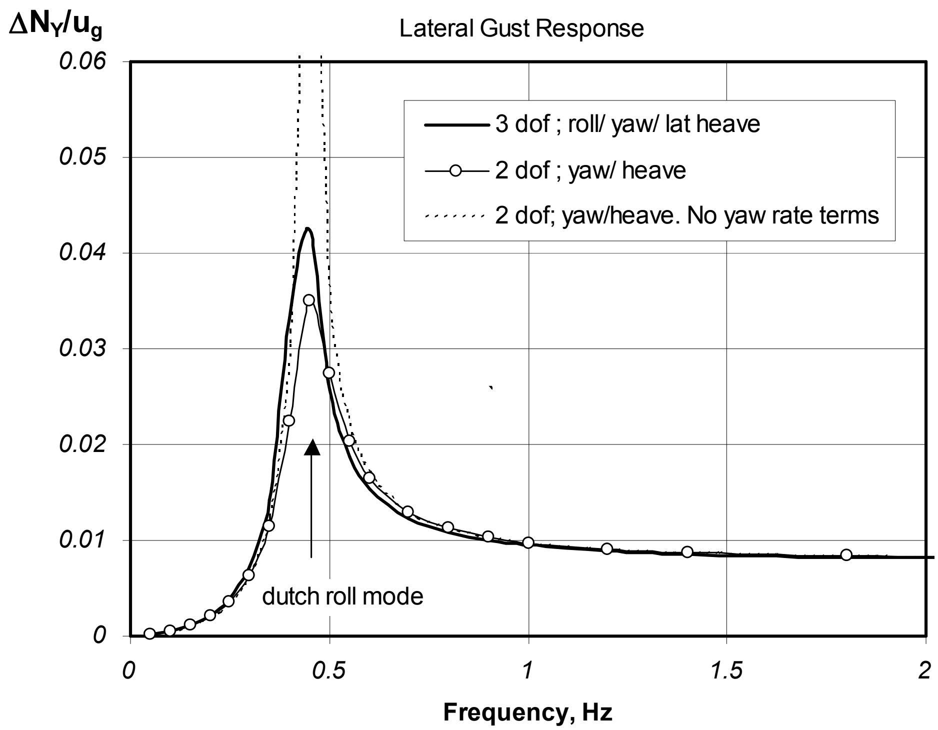


Table 1 Typical Operating Parameters (Cruise) for Various Airplane Categories

Class		Weight	Operating speed		Altitude	Wing chord	Wing span	Wing Area	Sweep
		Lb	Kt eas	mph, true	Ft.	Ft	Ft	Ft^2	deg
1	2-seat piston basic trainer	1,500	78	90	1,000	4.8	32.7	157	0
2	High performance piston single	2,750	137	170	5,000	5.4	33.4	180	0
3	Light piston twin	4,500	150	192	7,000	5.6	37.2	208	0
4	Light Air taxi, jet	5,000	230	382	23,000	4.1	35.6	144	0
5	Agricultural crop-duster	7,000	108	125	200	6.0	50.0	300	0
6	Pressurized exec twin-turboprop	9,800	175	258	16,000	5.8	50.3	294	0
7	Light high-performance business jet	12,000	275	538	32,000	4.5	42.4	191	29
8	Large business jet	25,000	240	506	36,000	7.3	51.2	374	22

Class		Pitch inertia	$C_{L\alpha}$	$C_{M\alpha}$	Wing loading	Aspect ratio	ρ/ρ_0	Kg	$\frac{Ve.C_{L\alpha}}{498(W/S)}$
		$Lb.ft^2$	rad^{-1}	rad^{-1}	Lb/ft^2			(note 1)	g's (note 2)
1	2-seat piston basic trainer	32,000	5.08	-0.761	9.6	6.8	0.971	0.585	0.0832
2	High performance piston single	87,000	5.09	-0.815	15.3	6.2	0.862	0.669	0.0971
3	Light piston twin	160,000	4.78	-0.670	21.7	6.7	0.811	0.732	0.0665
4	Light Air taxi, jet	210,000	5.63	-0.844	34.6	8.8	0.481	0.827	0.0751
5	Agricultural crop-duster	252,000	5.30	-0.794	23.3	8.3	0.994	0.690	0.0492
6	Pressurized exec twin-turboprop	479,000	5.33	-0.799	33.3	8.6	0.609	0.789	0.0562
7	Light high-performance business jet	1,050,000	5.18	-0.881	62.8	9.4	0.347	0.858	0.0456
8	Large business jet	2,770,000	5.51	-0.826	66.8	7.0	0.298	0.849	0.0397

Notes: (1) Kg is the discrete gust alleviation factor per the Pratt 1-dof approximation, from FAR 23.341
(2) Normal load factor for unit sharp-edge gust (u_{de} = 1 ft/sec equivalent) at the specified operating condition

Figure 6 Derived Vertical $+N_Z$ Spectra by Aircraft Category at Nominal Design Cruise Points (Using Mission Analysis)

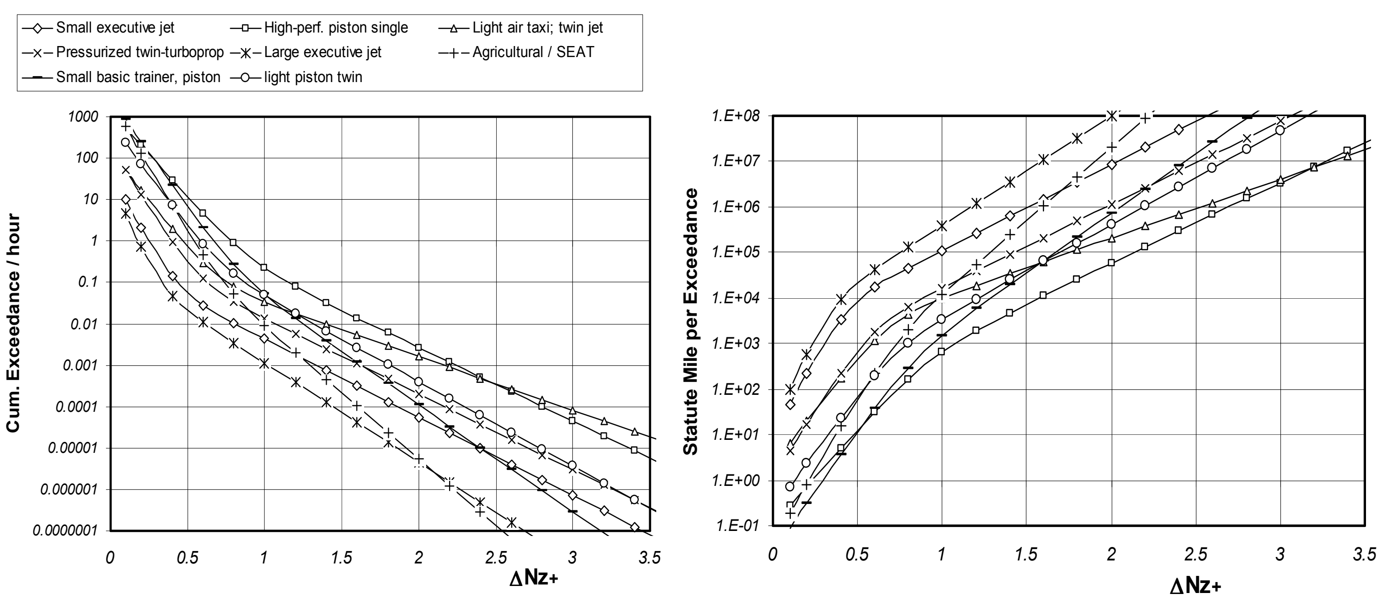


Figure 7 Comparison of Exceedance Curves for a High-Performance Business Jet

(a) High Altitude Cruise

(b) Terminal Area Operations

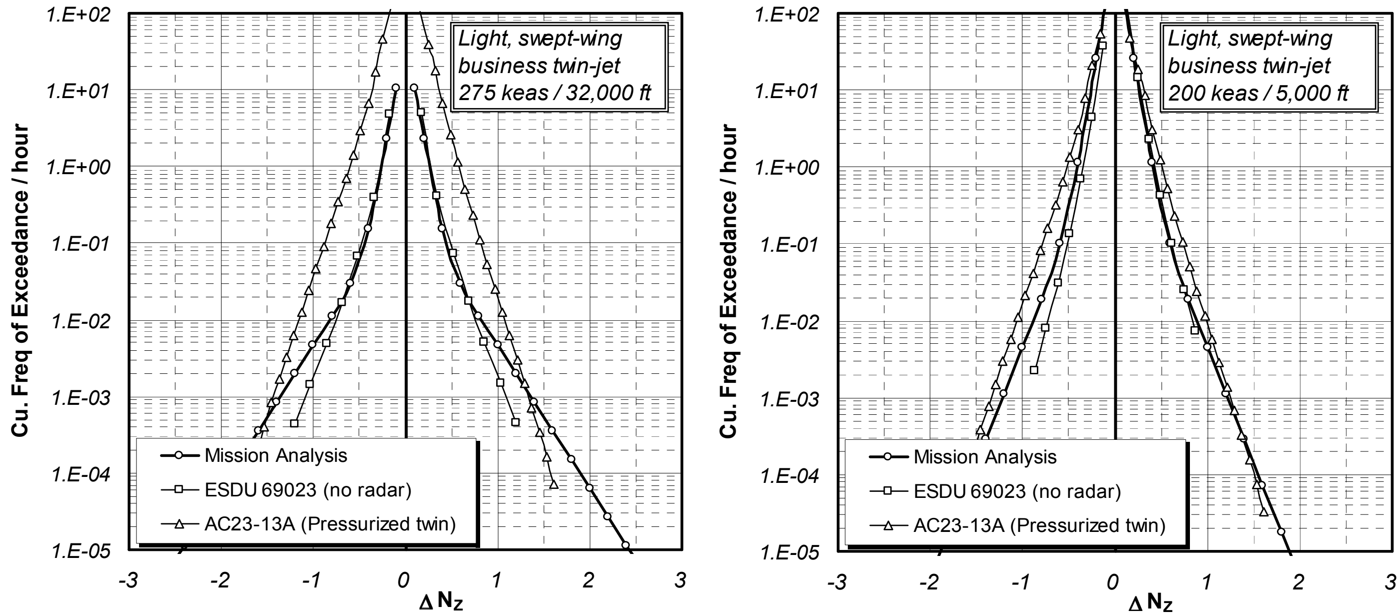


Figure 8 Agricultural Aircraft

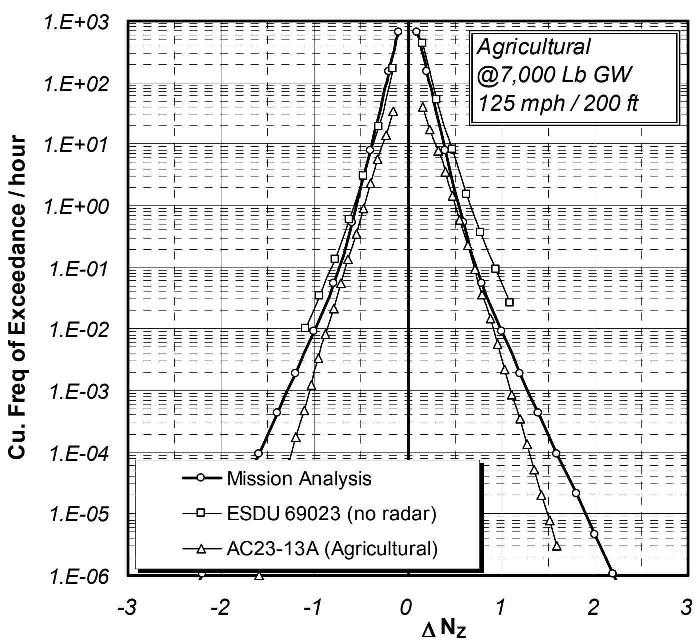


Figure 9 Single-Piston Light-Plane

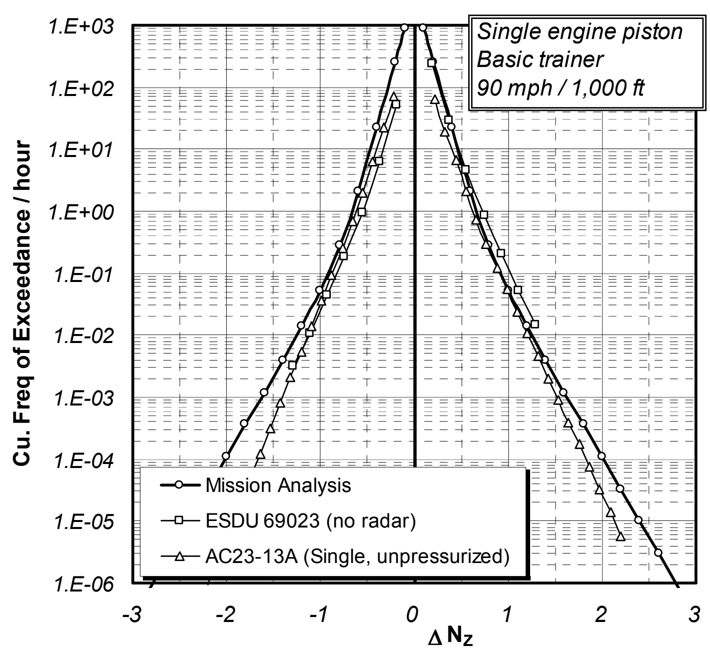
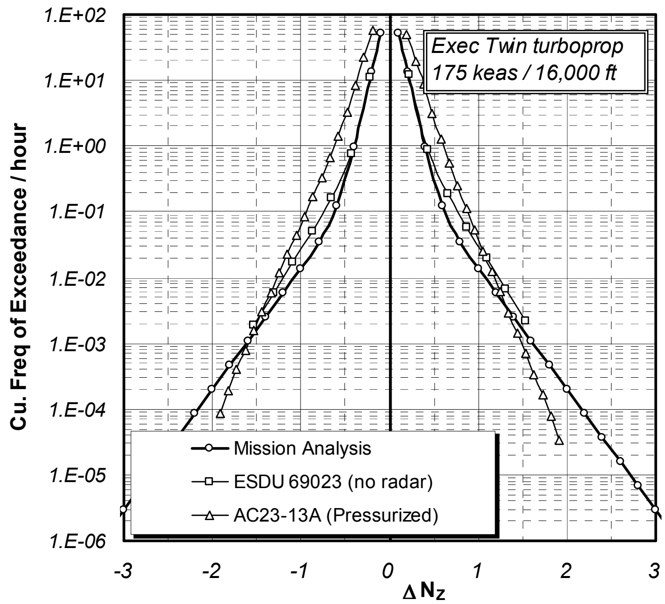
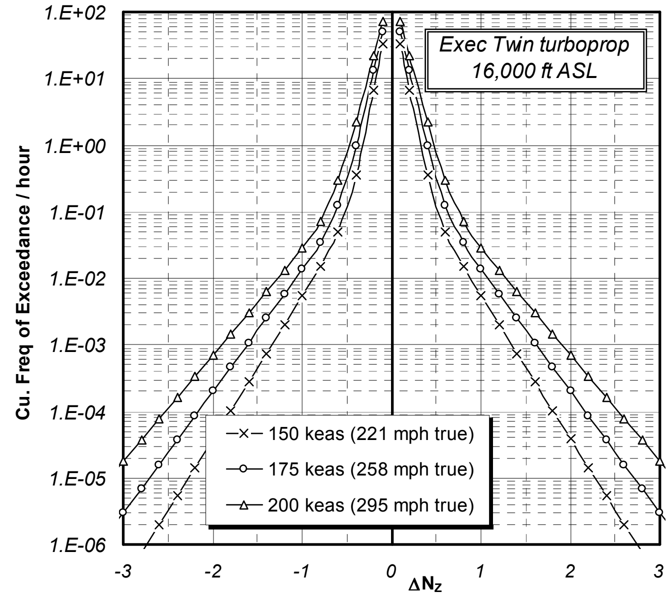


Figure 10 Effect of Operating Conditions on Gust Load Spectrum
Pressurized Executive Twin turboprop

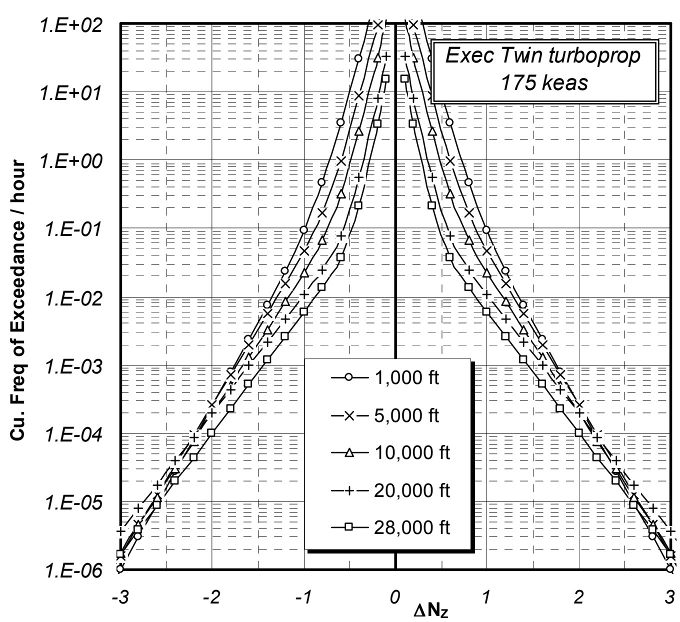
(a) Comparison of Methods at Nominal Cruise



(b) Effect of Airspeed



(c) Effect of Operating Altitude



(d) Effect of Aircraft Loading