SPRING-MASS-BUMPER SYSTEM SUBJECTED TO INITIAL EXICITATION

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Derivation

Consider a single-degree-of-freedom system as shown in Figure 1. This is a piecewise-linear system.

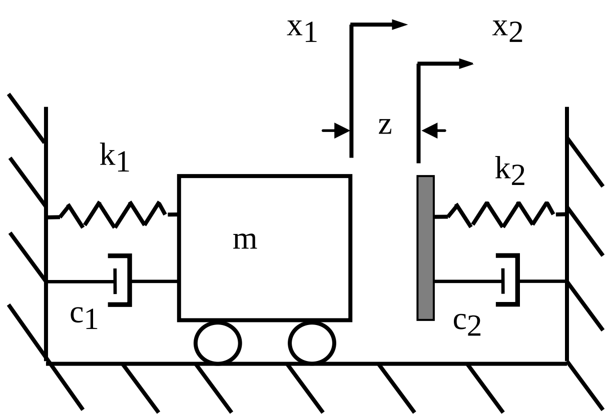

Figure 1.

The variables are

M	=	mass
c_1, c_2	=	damping coefficients
k_1, k_2	=	stiffness

Note that the relative displacement is

$$z = x_2 - x_1$$

Spring-Mass System

The equation of motion for the spring-mass system subjected to initial conditions is

$$m\ddot{x}_1 + c_1\dot{x}_1 + k_1 x_1 = 0\ ,\ \text{for } z > 0 \tag{1}$$

$$\ddot{x}_1 + \left(\frac{c_1}{m}\right)\dot{x}_1 + \left(\frac{k_1}{m}\right)x_1 = 0 \tag{2}$$

Now take the Laplace transform.

$$L\left\{\ddot{x}_1 + \left(\frac{c_1}{m}\right)\dot{x}_1 + \left(\frac{k_1}{m}\right)x_1\right\} = L\{0\} \tag{3}$$

$$\begin{aligned} &s^2 X_1(s) - s x_1(0) - \dot{x}_1(0) \\ &+ (c_1/m) s X_1(s) - (c_1/m) x_1(0) \\ &+ (k_1/m) X_1(s) \ = 0 \end{aligned} \tag{4}$$

$$\left\{s^2 + (c_1/m)s + (k_1/m)\right\}X_1(s) + \{-1\}\dot{x}_1(0) + \{-s - (c_1/m)\}x_1(0) = 0 \tag{5}$$

$$\left\{s^2 + (c_1/m)s + (k_1/m)\right\}X_1(s) = \dot{x}_1(0) + \{s + (c_1/m)\}x_1(0) \tag{6}$$

$$X_1(s) = \left\{\frac{\dot{x}_1(0) + \{s + (c_1/m)\}x_1(0)}{s^2 + (c_1/m)s + (k_1/m)}\right\} \tag{7}$$

$$X_1(s) = \left\{\frac{s\, x_1(0)}{s^2 + (c_1/m)s + (k_1/m)}\right\} + \left\{\frac{\dot{x}_1(0) + (c_1/m)x_1(0)}{s^2 + (c_1/m)s + (k_1/m)}\right\} \tag{8}$$

The inverse Laplace transform is

$$x_1(t) = \exp(-\beta t)\left\{ x(0)\cos(\alpha t) + \frac{2\left[m\dot{x}_1(0) + c_1 x_1(0)\right] - c_1 x_1(0)}{\sqrt{4k_1 m - c_1^2}}\sin(\alpha t)\right\} \tag{9}$$

where

$$\alpha = \frac{1}{2m}\sqrt{4k_1 m - c_1^2}$$

$$\beta = \frac{c_1}{2m}$$

The displacement equation simplifies to

$$x_1(t) = \exp(-\beta t)\left\{ x_1(0)\cos(\alpha t) + \frac{2m\dot{x}_1(0) + c_1 x_1(0)}{\sqrt{4k_1 m - c_1^2}}\sin(\alpha t)\right\} \quad , \text{ for } z > 0 \tag{10}$$

Bumper System

$$c_2\dot{x}_2 + k_2 x_2 = 0 \ , \text{ for } z > 0 \tag{11}$$

$$\dot{x}_2 + \left(\frac{k_2}{c_2}\right)x_2 = 0 \tag{12}$$

Now take the Laplace transform.

$$L\left\{ \dot{x}_2 + \left(\frac{k_2}{c_2} \right) x_2 \right\} = L\{0\} \tag{13}$$

$$sX_2(s) - x_2(0) + (k_2/c_2) X_2 = 0 \tag{14}$$

$$[s + (k_2/c_2)] X_2 - x_2(0) = 0 \tag{15}$$

$$X_2 = \frac{x_2(0)}{s + (k_2/c_2)} \tag{16}$$

The inverse Laplace transform is

$$x_2(t) = x_2(0) \exp(-\eta t) \quad , \text{ for } z > 0 \tag{17}$$

where

$$\eta = k_2/c_2$$

Note that the initial conditions need to be updated after as the mass contacts the bumper or as it is released. This also requires subtracting a delay term from the time variable.

Spring-Mass-Bumper System

The following formula is for the case then the mass and bumper are in contact.

$$x_1(t) = \exp\left(-\hat{\beta}t\right)\left\{x_1(0)\cos\left(\hat{\alpha}t\right) + \frac{2m\dot{x}_1(0) + \left(c_1 + c_2\right)x_1(0)}{\sqrt{4(k_1 + k_2)m - \left(c_1 + c_2\right)^2}}\sin\left(\hat{\alpha}t\right)\right\} \quad , \text{ for } \quad z=0 \tag{18}$$

where

$$\hat{\alpha} = \frac{1}{2m}\sqrt{4(k_1 + k_2)m - \left(c_1 + c_2\right)^2}$$

$$\ddot{\beta} = \frac{\left(c_1 + c_2\right)}{2m}$$

Reference

1. T. Irvine, Free Vibration of a Single-degree-of-freedom System, Revision B, Vibrationdata, 2005.

APPENDIX A

Example

The system in Figure 1 has the following parameters.

m	=	1 lbm
c_1	=	0.0163 lbf sec/in
k_1	=	10.22 lbf/in
c_2	=	0.0326 lbf sec/in
k_2	=	30.66 lbf/in

The initial conditions are

$\dot{x}_1(0)$	=	40 in/sec
$x_1(0)$	=	0 in
$x_2(0)$	=	0 in

$z = 0.2$ in when both systems are at rest.

The results are calculated using the formulas in the main text using Matlab script: SDOF_bumper.m.

The displacement and velocity are shown in Figures A-1 and A-2, respectively.

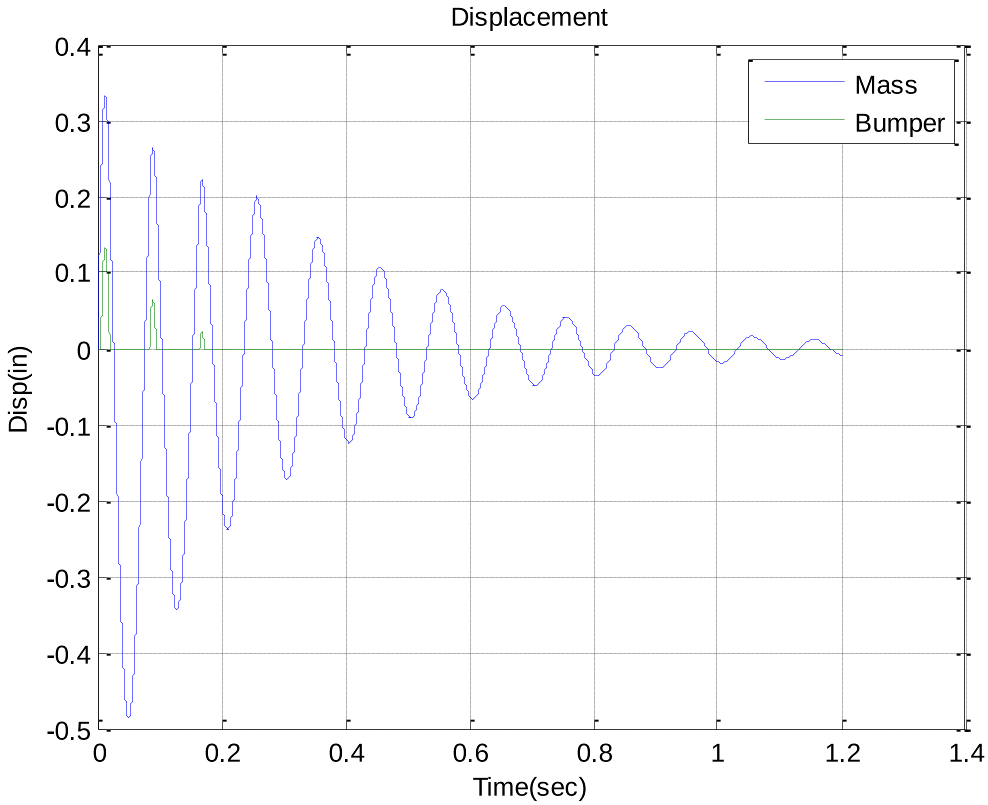

Figure A-1.

The mass contacts the bumper three times before 0.2 seconds.

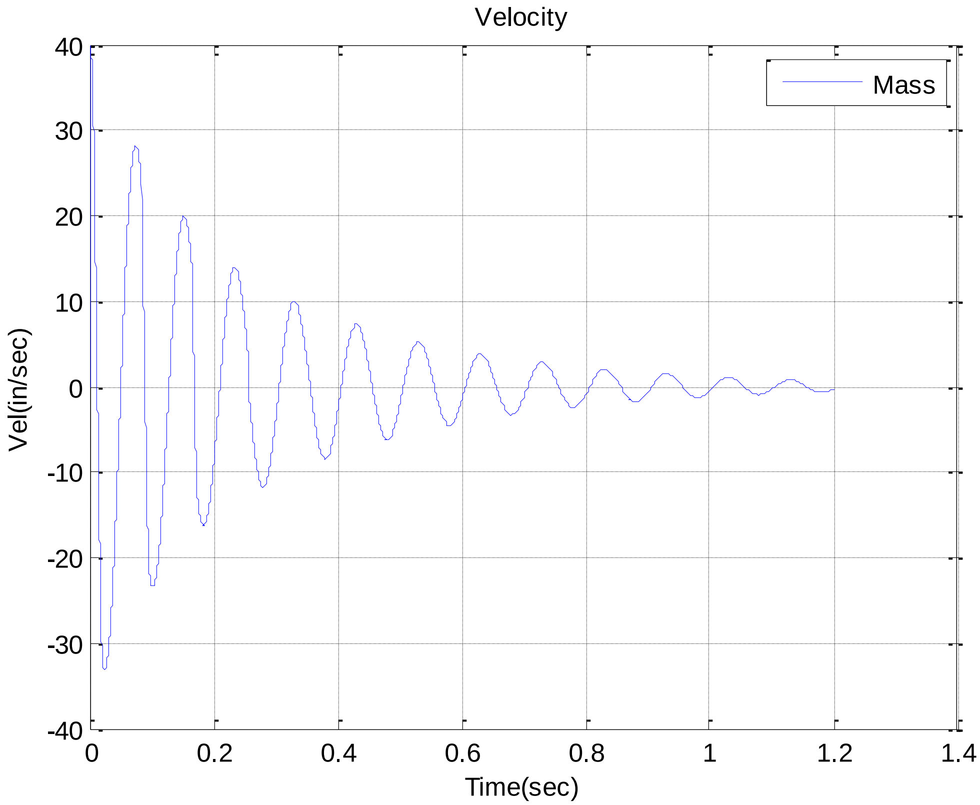
Velocity
Mass
Vel(in/sec)
Time(sec)
40
30
20
10
0
-10
-20
-30
-40
0
0.2
0.4
0.6
0.8
1
1.2
1.4

Figure A-2.