VIBRATION OF A SIMPLY SUPPORTED BEAM WITH CONCENTRATED MASS

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February 15, 2011

Consider a simply-supported beam with a discrete mass. Neglect rotary inertia and shear deformation.

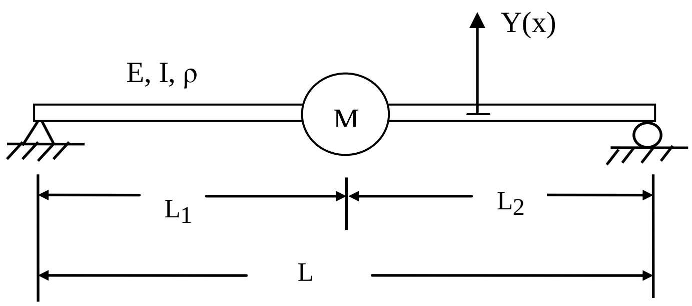

Figure 1.

E	is the modulus of elasticity
I	is the area moment of inertia
L	is the length
ρ	is the beam mass/length
M	is the discrete mass
Y	is the lateral displacement

Determine the natural frequencies using the Rayleigh-Ritz method from Reference 1.

The boundary conditions at the left end x = 0 are

$$Y(0) = 0 \qquad \text{(zero displacement)} \tag{1}$$

$$\left.\frac{d^2Y}{dx^2}\right|_{x=0} = 0 \qquad \text{(zero bending moment)} \tag{2}$$

The boundary conditions at the free end x = L are

$$Y(L) = 0 \qquad \text{(zero displacement)} \tag{3}$$

$$\left.\frac{d^2Y}{dx^2}\right|_{x=L} = 0 \qquad \text{(zero bending moment)} \tag{4}$$

$$\phi_1(x) = \sin\left(\frac{\pi x}{L}\right) \tag{5}$$

$$\phi_1'(x) = \left(\frac{\pi}{L}\right)\cos\left(\frac{\pi x}{L}\right) \tag{6}$$

$$\phi_1''(x) = -\left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\pi x}{L}\right) \tag{7}$$

$$\phi_2(x) = \sin\left(\frac{2\pi x}{L}\right) \tag{8}$$

$$\phi_2'(x) = \left(\frac{2\pi}{L}\right)\cos\left(\frac{2\pi x}{L}\right) \tag{9}$$

$$\phi_2''(x) = -\left(\frac{2\pi}{L}\right)^2 \sin\left(\frac{2\pi x}{L}\right) \tag{10}$$

$$\phi_3(x) = \sin\left(\frac{3\pi x}{L}\right) \tag{11}$$

$$\phi_3{}'(x) = \left(\frac{3\pi}{L}\right)\cos\left(\frac{3\pi x}{L}\right) \tag{12}$$

$$\phi_3{}''(x) = -\left(\frac{3\pi}{L}\right)^2 \sin\left(\frac{3\pi x}{L}\right) \tag{13}$$

The bending oscillation of a beam has the following stiffness and mass coefficients

$$k_{ij} = \int EI\,\phi_i{}''\,\phi_j{}''\,dx \tag{14}$$

$$m_{ij} = M\,\phi_i(L_1)\phi_j(L_1) + \int \rho\,\phi_i\,\phi_j\,dx \tag{15}$$

$$m_{11} = M\,\phi_1(L_1)\phi_1(L_1) + \int \rho\,\phi_1\,\phi_1\,dx \tag{16}$$

$$m_{11} = M[\phi_1(L_1)]^2 + \int \rho\,\phi_1{}^2\,dx \tag{17}$$

$$m_{11} = M\left[\sin\left(\frac{\pi L_1}{L}\right)\right]^2 + \rho\int\left[\sin\left(\frac{\pi x}{L}\right)\right]^2 dx \tag{18}$$

$$m_{11} = M\left[\sin\left(\frac{\pi L_1}{L}\right)\right]^2 + \frac{\rho}{2}\int\left[1-\cos\left(\frac{2\pi x}{L}\right)\right] dx \tag{19}$$

$$m_{11} = M\left[\sin\left(\frac{\pi L_1}{L}\right)\right]^2 + \frac{\rho}{2}\left[x-\left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi x}{L}\right)\right]\Bigg|_0^L \tag{20}$$

$$m_{11} = M\left[\sin\left(\frac{\pi L_1}{L}\right)\right]^2 + \frac{\rho L}{2} \tag{21}$$

$$m_{12} = M\phi_1(L_1)\phi_2(L_1) + \int \rho\, \phi_1\, \phi_2\, dx \tag{22}$$

$$m_{12} = M\sin\left(\frac{\pi L_1}{L}\right)\sin\left(\frac{2\pi L_1}{L}\right) + \int \rho \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)dx \tag{23}$$

$$m_{12} = M\sin\left(\frac{\pi L_1}{L}\right)\sin\left(\frac{2\pi L_1}{L}\right) + \frac{\rho}{2}\int \left[-\cos\left(\frac{3\pi x}{L}\right) + \cos\left(\frac{\pi x}{L}\right)\right]dx \tag{24}$$

$$m_{12} = M\sin\left(\frac{\pi L_1}{L}\right)\sin\left(\frac{2\pi L_1}{L}\right) + \frac{\rho}{2}\left[-\left(\frac{L}{3\pi}\right)\sin\left(\frac{3\pi x}{L}\right) + \left(\frac{L}{\pi}\right)\sin\left(\frac{\pi x}{L}\right)\right]\Bigg|_0^L \tag{25}$$

$$m_{12} = M\sin\left(\frac{\pi L_1}{L}\right)\sin\left(\frac{2\pi L_1}{L}\right) \tag{26}$$

$$m_{13} = M\phi_1(L_1)\phi_3(L_1) + \int \rho\, \phi_1\, \phi_2\, dx \tag{27}$$

$$m_{13} = M\sin\left(\frac{\pi L_1}{L}\right)\sin\left(\frac{3\pi L_1}{L}\right) + \int \rho \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{3\pi x}{L}\right)dx \tag{28}$$

$$m_{13} = M\sin\left(\frac{\pi L_1}{L}\right)\sin\left(\frac{3\pi L_1}{L}\right) + \frac{\rho}{2}\int \left[-\cos\left(\frac{4\pi x}{L}\right) + \cos\left(\frac{2\pi x}{L}\right)\right]dx \tag{29}$$

$$m_{13} = M\sin\left(\frac{\pi L_1}{L}\right)\sin\left(\frac{3\pi L_1}{L}\right) + \frac{\rho}{2}\left[-\left(\frac{L}{4\pi}\right)\sin\left(\frac{4\pi x}{L}\right) + \left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi x}{L}\right)\right]\Bigg|_0^L \quad (30)$$

$$m_{13} = M\sin\left(\frac{\pi L_1}{L}\right)\sin\left(\frac{3\pi L_1}{L}\right) \quad (31)$$

$$m_{22} = M\phi_2(L_1)\phi_2(L_1) + \int \rho\,\phi_2\,\phi_2\,dx \quad (32)$$

$$m_{22} = M[\phi_2(L_1)]^2 + \int \rho\,\phi_2{}^2\,dx \quad (33)$$

$$m_{22} = M\left[\sin\left(\frac{2\pi L_1}{L}\right)\right]^2 + \rho\int\left[\sin\left(\frac{2\pi x}{L}\right)\right]^2 dx \quad (34)$$

$$m_{22} = M\left[\sin\left(\frac{2\pi L_1}{L}\right)\right]^2 + \frac{\rho}{2}\int\left[1 - \cos\left(\frac{4\pi x}{L}\right)\right] dx \quad (35)$$

$$m_{22} = M\left[\sin\left(\frac{2\pi L_1}{L}\right)\right]^2 + \frac{\rho}{2}\left[x - \left(\frac{L}{4\pi}\right)\sin\left(\frac{4\pi x}{L}\right)\right]\Bigg|_0^L \quad (36)$$

$$m_{22} = M\left[\sin\left(\frac{2\pi L_1}{L}\right)\right]^2 + \frac{\rho L}{2} \quad (37)$$

$$m_{23} = M\phi_2(L_1)\phi_3(L_1) + \int \rho\, \phi_2\, \phi_3\, dx \tag{38}$$

$$m_{23} = M\sin\left(\frac{2\pi L_1}{L}\right)\sin\left(\frac{3\pi L_1}{L}\right) + \int \rho\,\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{3\pi x}{L}\right)dx \tag{39}$$

$$m_{23} = M\sin\left(\frac{2\pi L_1}{L}\right)\sin\left(\frac{3\pi L_1}{L}\right) + \frac{\rho}{2}\int \left[-\cos\left(\frac{5\pi x}{L}\right) + \cos\left(\frac{\pi x}{L}\right)\right]dx \tag{40}$$

$$m_{23} = M\sin\left(\frac{2\pi L_1}{L}\right)\sin\left(\frac{3\pi L_1}{L}\right) + \frac{\rho}{2}\left[-\left(\frac{L}{5\pi}\right)\sin\left(\frac{5\pi x}{L}\right) + \left(\frac{L}{\pi}\right)\sin\left(\frac{\pi x}{L}\right)\right]\Bigg|_0^L \tag{41}$$

$$m_{23} = M\sin\left(\frac{2\pi L_1}{L}\right)\sin\left(\frac{3\pi L_1}{L}\right) \tag{42}$$

$$m_{33} = M\phi_3(L_1)\phi_3(L_1) + \int \rho\, \phi_3\, \phi_3\, dx \tag{43}$$

$$m_{33} = M[\phi_3(L_1)]^2 + \int \rho\, \phi_3{}^2\, dx \tag{44}$$

$$m_{33} = M\left[\sin\left(\frac{3\pi L_1}{L}\right)\right]^2 + \rho\int \left[\sin\left(\frac{3\pi x}{L}\right)\right]^2 dx \tag{45}$$

$$m_{33} = M\left[\sin\left(\frac{3\pi L_1}{L}\right)\right]^2 + \frac{\rho}{2}\int \left[1 - \cos\left(\frac{6\pi x}{L}\right)\right] dx \tag{46}$$

$$m_{33} = M\left[\sin\left(\frac{3\pi L_1}{L}\right)\right]^2 + \frac{\rho}{2}\left[x - \left(\frac{L}{6\pi}\right)\sin\left(\frac{6\pi x}{L}\right)\right]\Bigg|_0^L \tag{47}$$

$$m_{33} = M\left[\sin\left(\frac{3\pi L_1}{L}\right)\right]^2 + \frac{\rho L}{2} \tag{48}$$

$$k_{ij} = \int EI\,\phi_i''\,\phi_j''\,dx \tag{49}$$

$$\phi_1''(x) = -\left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\pi x}{L}\right) \tag{50}$$

$$k_{11} = \int EI\,\phi_1''\,\phi_1''\,dx \tag{51}$$

$$k_{11} = EI\left(\frac{\pi}{L}\right)^4 \int \left[\sin\left(\frac{\pi x}{L}\right)\right]^2 dx \tag{52}$$

$$k_{11} = \frac{1}{2} EI\left(\frac{\pi}{L}\right)^4 \int \left[1 - \cos\left(\frac{2\pi x}{L}\right)\right] dx \tag{53}$$

$$k_{11} = \frac{1}{2} EI\left(\frac{\pi}{L}\right)^4 \int \left[1 - \cos\left(\frac{2\pi x}{L}\right)\right] dx \tag{54}$$

$$k_{11} = \frac{1}{2} EI\left(\frac{\pi}{L}\right)^4 \left[x - \left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi x}{L}\right)\right]\Bigg|_0^L \tag{55}$$

$$k_{11} = \frac{1}{2} EI\left(\frac{\pi}{L}\right)^4 L \tag{56}$$

$$k_{12} = \int EI\,\phi_1''\,\phi_2''\,dx \tag{57}$$

$$k_{12} = EI\left(\frac{\pi}{L}\right)^2\left(\frac{2\pi}{L}\right)^2\int\left[\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\right]dx \tag{58}$$

$$k_{12} = \frac{1}{2}EI\left(\frac{\pi}{L}\right)^2\left(\frac{2\pi}{L}\right)^2\int\left[-\cos\left(\frac{3\pi x}{L}\right)+\cos\left(\frac{\pi x}{L}\right)\right]dx \tag{59}$$

$$k_{12} = \frac{1}{2}EI\left(\frac{\pi}{L}\right)^2\left(\frac{2\pi}{L}\right)^2\left[-\left(\frac{L}{3\pi}\right)\sin\left(\frac{3\pi x}{L}\right)+\left(\frac{L}{\pi}\right)\sin\left(\frac{\pi x}{L}\right)\right]\Bigg|_0^L \tag{60}$$

$$k_{12} = 0 \tag{61}$$

$$k_{13} = \int EI\,\phi_1''\,\phi_3''\,dx \tag{62}$$

$$k_{13} = EI\left(\frac{\pi}{L}\right)^2\left(\frac{3\pi}{L}\right)^2\int\left[\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{3\pi x}{L}\right)\right]dx \tag{63}$$

$$k_{13} = \frac{1}{2}EI\left(\frac{\pi}{L}\right)^2\left(\frac{3\pi}{L}\right)^2\int\left[-\cos\left(\frac{4\pi x}{L}\right)+\cos\left(\frac{2\pi x}{L}\right)\right]dx \tag{64}$$

$$k_{13} = \frac{1}{2}EI\left(\frac{\pi}{L}\right)^2\left(\frac{3\pi}{L}\right)^2\left[-\left(\frac{L}{4\pi}\right)\sin\left(\frac{4\pi x}{L}\right)+\left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi x}{L}\right)\right]\Bigg|_0^L \tag{65}$$

$$k_{13} = 0 \tag{66}$$

$$k_{22} = \int EI\,\phi_2''\,\phi_2''\,dx \tag{67}$$

$$k_{22} = EI\left(\frac{2\pi}{L}\right)^4 \int \left[\sin\left(\frac{2\pi x}{L}\right)\right]^2 dx \qquad (68)$$

$$k_{22} = \frac{1}{2} EI\left(\frac{2\pi}{L}\right)^4 \int \left[1 - \cos\left(\frac{4\pi x}{L}\right)\right] dx \qquad (69)$$

$$k_{22} = \frac{1}{2} EI\left(\frac{2\pi}{L}\right)^4 \int \left[1 - \cos\left(\frac{4\pi x}{L}\right)\right] dx \qquad (70)$$

$$k_{22} = \frac{1}{2} EI\left(\frac{2\pi}{L}\right)^4 \left[x - \left(\frac{L}{4\pi}\right)\sin\left(\frac{4\pi x}{L}\right)\right]\Bigg|_0^L \qquad (71)$$

$$k_{22} = \frac{1}{2} EI\left(\frac{2\pi}{L}\right)^4 L \qquad (72)$$

$$k_{23} = \int EI \phi_2'' \phi_3'' dx \qquad (73)$$

$$k_{23} = EI\left(\frac{2\pi}{L}\right)^2 \left(\frac{3\pi}{L}\right)^2 \int \left[\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{3\pi x}{L}\right)\right] dx \qquad (74)$$

$$k_{23} = \frac{1}{2} EI\left(\frac{2\pi}{L}\right)^2 \left(\frac{3\pi}{L}\right)^2 \int \left[-\cos\left(\frac{5\pi x}{L}\right) + \cos\left(\frac{\pi x}{L}\right)\right] dx \qquad (75)$$

$$k_{23} = \frac{1}{2} EI\left(\frac{2\pi}{L}\right)^2 \left(\frac{3\pi}{L}\right)^2 \left[-\left(\frac{L}{5\pi}\right)\sin\left(\frac{5\pi x}{L}\right) + \left(\frac{L}{\pi}\right)\sin\left(\frac{\pi x}{L}\right)\right]\Bigg|_0^L \qquad (76)$$

$$k_{23} = 0 \qquad (77)$$

$$k_{33} = \int EI\,\phi_3''\,\phi_3''\,dx \tag{78}$$

$$k_{33} = EI\left(\frac{3\pi}{L}\right)^4 \int \left[\sin\left(\frac{3\pi x}{L}\right)\right]^2 dx \tag{79}$$

$$k_{33} = \frac{1}{2} EI\left(\frac{3\pi}{L}\right)^4 \int \left[1 - \cos\left(\frac{6\pi x}{L}\right)\right] dx \tag{80}$$

$$k_{33} = \frac{1}{2} EI\left(\frac{3\pi}{L}\right)^4 \int \left[1 - \cos\left(\frac{6\pi x}{L}\right)\right] dx \tag{81}$$

$$k_{33} = \frac{1}{2} EI\left(\frac{3\pi}{L}\right)^4 \left[x - \left(\frac{L}{6\pi}\right)\sin\left(\frac{6\pi x}{L}\right)\right]\Bigg|_0^L \tag{82}$$

$$k_{33} = \frac{1}{2} EI\left(\frac{3\pi}{L}\right)^4 L \tag{83}$$

The generalized eigenvalue problem is

$$\begin{bmatrix} k_{11} - \omega^2 m_{11} & k_{12} - \omega^2 m_{12} & k_{13} - \omega^2 m_{13} \\ & k_{22} - \omega^2 m_{22} & k_{23} - \omega^2 m_{23} \\ & & k_{33} - \omega^2 m_{33} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{84}$$

The coefficient matrix is shown in upper triangular form due to symmetry.

The natural frequencies are

$$f_n = \frac{1}{2\pi}\omega_n \tag{85}$$

Reference

1. T. Irvine, Rayleigh-Ritz Methods for Beams and Rods, Vibrationdata, 1999.

APPENDIX A

The equations in the main text are implemented in a Matlab script.

Example 1

```
>> ss_beam_mass

 ss_beam_mass.m   ver 1.0  Feb 15, 2011

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 This program calculates natural frequencies of
 a simply-supported beam with a discrete mass via
 Rayleigh-Ritz Method.


 Enter the total length (in) 24

 Enter the length to the discrete mass (in) 0

 Enter the cross-section:
 1=rectangle  2=solid cylinder  3=other 2

 Enter the diameter (inch) 1

 Enter material:
 1=aluminum  2=steel  3=other 1

 Enter the discrete mass (lbm) 0

  Beam Mass =    1.885 lbm
 Total Mass =    1.885 lbm

   n     fn(Hz)
   1     133.9
   2     535.8
   3      1206
```

Example 2

```
>> ss_beam_mass

 ss_beam_mass.m   ver 1.0  Feb 15, 2011

 by Tom Irvine  Email: tomirvine@aol.com

 This program calculates natural frequencies of
 a simply-supported beam with a discrete mass via
 Rayleigh-Ritz Method.


 Enter the total length (in) 24

 Enter the length to the discrete mass (in) 12

 Enter the cross-section:
 1=rectangle  2=solid cylinder  3=other 2

 Enter the diameter (inch) 1

 Enter material:
 1=aluminum  2=steel  3=other 1

 Enter the discrete mass (lbm) 2

  Beam Mass =    1.885 lbm
 Total Mass =    3.885 lbm

   n     fn(Hz)
   1     75.59
   2     535.8
   3     932.8
```